CONTRIBUTIONS
TO
TWO-DIMENSIONAL INCLUSION PROBLEM IN ELASTICITY

by

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CERTIFICATE

This is to certify that the thesis entitled 'Contributions to Two-Dimensional Inclusion Problem' that is being submitted by Shri H. C. Radha Krishna, B.E., M.Sc., for the award of the Degree of Doctor of Philosophy to the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision and guidance. Shri H. C. Radha Krishna has worked for two years in the Indian Institute of Technology, Kanpur and the thesis has reached the standard fulfilling the requirements of the regulations to the Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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CONTENTS

Short List of Symbols

x, y, z	Co-ordinate axes
$u_i \ (i = x, y, z)$	Displacement Components
$\varepsilon_{ij}, e_{ij} \ (i, j = x, y, z)$	Strain Components
P_{ij}, p_{ij}	Stress Components
E	Young's modulus
ν	Poisson's Ratio
$K = \dfrac{E}{3(1-2\nu)}$	Bulk modulus
$\lambda = \nu E/(1+\nu)(1-2\nu)$	Lame's Constant
$\mu = E/2(1+\nu)$	Rigidity modulus
$\alpha_{11}, \alpha_{22}, \alpha_{12}, \alpha_{66}$	Elastic Constants of Orthotropic Matrix Material
$\beta_{11}, \beta_{22}, \beta_{12}, \beta_{66}$	Elastic Constants of Orthotropic Inclusion Material

INTRODUCTION

A class of inclusion problem was first studied by Mott and Nabarro in connection with their theory of precipitation hardening of alloys in 1940 and by Frenkel in 1946 in connection with his theory of liquids. No further progress was made since then, because the methods that were employed were useful only for highly symmetrical figures like those of a circle or a sphere. In 1957 Eshelby introduced the point-force concept of solving inclusion problems. Although he obtained some general theorems of great theoretical interest, no explicit solution to any problem was given. However in a subsequent paper in 1959, he obtained some solutions relating to ellipsoidal inclusions. Bhargava in 1959 and Jaswon and Bhargava in 1961 made substantial progress in obtaining solutions for some two-dimensional problems by introducing the complex variable technique coupled with point-force concept. The work done in this thesis is the contribution to the subject beyond this stage.

In all these problems inclusion is a region in an infinite elastic medium tending to undergo a change in its form. The constraints imposed by its surroundings, called matrix, develop a system of locked-up accommodation stresses in both of them. Determination of the elastic field in the matrix and in the inclusion forms an important problem in elasticity theory.

In the works cited above, the inclusion is of the same material as that of the matrix. Nabarro solved the problem when the inclusion is of a cylindrical or spherical shape. His method consists in finding that interface at which the normal stresses are continuous. As already pointed out this method would be successful only in circular and spherical case because the equilibrium boundary would be circular or spherical on physical grounds.

In Eshelby's paper, the method consists in making certain imaginary operations allowing the inclusion to expand, then reducing it to the size of the hole and finally building a system of forces, which he calls a layer of point forces. The cumulative effect of these forces is found by integration. A method is suggested to deal with the case when the inclusion is of a different material, called the inhomogeneity. The point-force method itself is a cumbersome one and to obtain explicit solution when an otherwise uniform stress field is present, is a complicated work.

However, it was felt that well-known energy principle which has been successfully employed for finding the exact or approximate solutions to a variety of problems, can be employed to this class of problems. This has been done in this thesis with considerable success. It was surprisingly revealed that not only it solves the problem in a straight forward manner, but also gives greater freedom in choosing the elastic constants of materials concerned. As a first example the spherical and cylindrical inclusions are solved. The results check up with those of Nabarro.

As a next example the case of elliptic inclusions has been considered. Apart from the fact that the elastic properties of the materials are different, the inclusion itself can undergo a principal or shear strain. These are actually two different cases and should be considered as such, in case of principal strains, the axes of the elliptic inclusions before and after deformation and in equilibrium position, remain coincident. In case of shear strains, the axes rotate. This latter case has greater applications to technological processes.

The equilibrium boundary, the stress-strain field, and the energy in the matrix and in the inclusion are determined. In obtaining these results, complex variable techniques, semi-inverse method and Cleyparon's theorem for energy determination, have been widely used along with the minimum energy principle. The results in each case were substantiated by proving the continuity of the normal and shear stresses at the interface.

It is interesting to find how these results are modified when an external force field is applied to the matrix at infinity. As a simple example first the solution of the inhomogeneity problem of a spherical region in a stressed matrix is considered. Then the treatment is extended to discuss the case of a cylindrical inhomogeneity.

The problem is then extended to the case of an elliptic inhomogeneity in an isotropic medium under tension at large distances from the inhomogeneity. Again exact analytical solutions have been obtained. This solution is very general and many particular cases such as uniaxial tension or compression

or pure shear at infinity can be deduced. Further the result for the case of a cavity can be readily obtained by making the elastic constants of the inclusion tend to zero.

In determining the solutions of the above cases, the materials of the inclusion and matrix have been assumed to be isotropic. Solution for the case when the inhomogeneity is having cubic or orthotropic properties, had not been obtained so far. In fact not many solutions for the first or the second fundamental problem for orthotropic elastic materials are known at present. In this thesis the inclusion problem in orthotropic elasticity has been considered for the first time. The problem involves the solution of the second boundary value problem of an infinite region with a hole. Explicit solutions for the circular and elliptic inclusions have been obtained. It may be remarked that great care than that necessary for isotropic bodies has to be exercised in assuming the equilibrium boundary.

The engineering application of the theory of cavities to the calculation of stress concentrations has been discussed by Neuber. The results obtained in this thesis can all be used in this. These results find a use in the theory of martensitic transformations. The transformation strain for martensitic inclusions is essentially a pure shear. But it will be dilation if the inclusion is a particle of precipitate formed by diffusion. The results of the inhomogeneity problem when the matrix is under stress simplifies the determination of the bulk elastic constants of inhomogeneous elastic aggreggates. These results also find an application in many design problems, for example the design of a underground multilayer reservoir for

high pressure gas, the determination of stresses due to insertion of a screw or a rivet etc.

If the inclusion is subjected to some twisting moment, the problem would involve the solution of the well known Laplace's equation. The solution of such a problem can be obtained by relaxation method or the electrical analogue methods. However, a method based on Green's theorem is stated and has been extensively checked with reference to the torsion problem of a rectangular prism for which analytical solutions are available. It appears, the method would be extremely useful when digital computers are available.

Thanks are due to Prof. J. G. Oldroyd for having communicated the papers entitled '(1) Two-Dimensional Elliptic Inclusion, (2) Elliptic Inclusion in Stressed Matrix, (3) Elliptic Inclusion in Orthotropic Medium' to the Proceedings of Cambridge Philosophical Society, and Prof. P. L. Bhatnagar, for having communicated the paper ' Numerical Solution of Two-Dimensional Laplace's Equation ' to proceedings of National Institute of Sciences.

PART I

CHAPTER I

Spherical Inclusion

Consider a sphere of radius a , termed an inclusion, embedded in an infinite continuous isotropic unstressed elastic medium which would undergo a dimensional change to a concentric sphere of radius $a(1+\delta)$ in the absence of the surrounding material, the matrix. The surface of the inclusion in the absence of the matrix will be termed a free surface. The inclusion can undergo this non-elastic displacement due to various reasons such as non-uniform heating, plastic deformation etc.

An alternate model of the inclusion would be as follows. Suppose there is a spherical cavity of radius a in an infinite continuous isotropic unstressed elastic material. A sphere of radius $a(1+\delta)$ is introduced into this cavity. Obviously this sphere has a free surface of radius $a(1+\delta)$.

Because of the presence of the surrounding material, the inclusion would not be able to attain its free-surface configuration in our first model. In the case of the second model if continuity of the inclusion and the matrix materials is to be maintained, the inclusion would not be able to retain its original size. In both cases a system of locked-up accommodation stresses would be generated. Determination of the stress field in both the inclusion and the matrix, the equilibrium boundary and the related problems form the subject matter of this thesis.

The two models presented in the first two preceding paragraphs are mathematically equivalent so far as calculation of elastic field is concerned. In first model, the inclusion is

strained when it tends to attain its free-surface configuration. However, this is non-elastic strain and is not related to any stress field by Hooke's Law. In the second model, no such strain exists. However, in both models of the inclusion, the elastic strain which is related to stresses is calculated from the displacement field from the free-surface. These are the same if the materials of the two models are the same.

We shall be concerned with infinitesimal strain throughout this work, and hence it is assumed that δ falls within its limits. Further we assume that no relative slipping takes place in the process of the non-elastic displacement in the case of the first model or while introducing the inclusion in the case of the second model. Also the continuity of inclusion and matrix is always maintained.

Inclusion problems were studied about twenty years back by Mott and Nabarro [1, 2] in connection with their theory of precipitation hardening of alloys, and by Frenkel [3] in connection with his theory of liquids. However, Eshelby [4] made a systematic investigation in 1957 by introducing the point-force concept. Jaswon and Bhargava [5] extended the method by introducing complex variable technique in conjunction with point-force concept to find explicit solutions for two-dimensional problems.

Nabarro's method can be explained as follows. Let the radius of the inclusion in the equilibrium position be $a(1+\epsilon)$ Hence the displacement at the internal boundary of the matrix is $a\epsilon$. The radial elastic displacement on the boundary of the inclusion would be $-a(\delta-\epsilon)$. Let P be the common

pressure in the equilibrium position. This would be the same at all points, because of the complete symmetry of the inclusion and matrix. The configuration is illustrated in fig. 1 where thick line represent the free-surface of the inclusion, the thin line the initial boundary of the matrix and the dotted line show the equilibrium position.

The radial, hoop and shear strains $\mathcal{E}_{rr}$, $\mathcal{E}_{\theta\theta}$, $\mathcal{E}_{r\theta}$ and also corresponding stresses P_{rr}, $P_{\theta\theta}$, $P_{r\theta}$ throughout the inclusion due to the uniform pressure P are [6]

$$\mathcal{E}_{rr} = \mathcal{E}_{\theta\theta}\ ,\ \mathcal{E}_{r\theta} = 0\ ,\quad P_{rr} = P_{\theta\theta} = -P\ ,\quad P_{r\theta} = 0$$

and the dilation field is

$$\Delta = 3\,\mathcal{E}_{rr}\ .$$

For ease of exposition small letters have been used for the displacement, strains, stresses etc. for the matrix, unless otherwise stated and capital letters for the inclusion.

By Hooke's law relating dilation to the mean hydrostatic stress

$$\Delta = \frac{P_{rr} + P_{\theta\theta} + P_{\theta\theta}}{3K}\ ,\quad 3\mathcal{E}_{rr} = -\frac{P}{K}\ ,$$

where K is the bulk modulus of the inclusion material. Noting that $\mathcal{E}_{\theta\theta} = U_r / r$ where U_r is the radial displacement, we have

$$U_r = -\frac{Pr}{3K}$$

whence the radial displacement at its boundary $r = a(1+\delta)$ will be

$$-\frac{pa(1+\delta)}{3K}$$

Therefore, to the first order of approximation

$$\delta - \epsilon = \frac{p}{3K} \tag{1}$$

As regards the matrix for a uniform pressure at the inner boundary, the radial, hoop and shear stresses p_{rr}, $p_{\theta\theta}$, $p_{r\theta}$ are respectively given by [6]

$$p_{rr} = \frac{C}{r^3}, \qquad p_{\theta\theta} = -\frac{C}{2r^3}, \qquad p_{r\theta} = 0$$

where the constant C is determined by the condition that

$$(p_{rr})_{r=a} = -p, \quad \text{whence} \quad C = -pa^3 .$$

By stress-strain relations

$$e_{\theta\theta} = \frac{u_r}{r} = \frac{(1+\nu)pa^3}{2Er^3}$$

where E is the Young's modulus and ν the Poisson's ratio of the matrix material. Hence

$$u_r = \frac{(1+\nu)pa^3}{2Er^2}$$

Therefore u_r at $r = a$ is $(1+\nu)\,pa/2E$ which is equal to $a\epsilon$. Therefore

$$\epsilon = \frac{p}{4\mu}, \tag{2}$$

where μ is the shear modulus of matrix material and is equal to $E/2(1+\nu)$. Equating the value of p from equations (1) and (2) we get

$$3K(\delta-\epsilon) = 4\mu\epsilon ,$$

whence

$$\epsilon = \frac{3K\delta}{3K+4\mu} , \tag{3}$$

and the equilibrium pressure by (2) is

$$p = \frac{12K\mu\delta}{3K+4\mu} \tag{4}$$

The strain energy in the inclusion and matrix will be

$$W_I = 96\pi a^3 K \frac{\mu^2\delta^2}{(3K+4\mu)^2} , \qquad W_m = 72\pi a^3\mu \frac{K^2\delta^2}{(3K+4\mu)^2} \tag{5}$$

This chapter explains the type of problems and the final results with which this thesis is concerned. To provide a unified picture, the basic equations of elasticity are very briefly given for general case in second chapter and for the plane case in the third chapter.

CHAPTER II

Equations of Elasticity Theory

The stress field at a point in the body can be described in terms of 9 stress components p_{ij} ($i, j = x, y, z$ in the cartesian coordinate system). By consideration of the equilibrium of an element it can be readily established that $p_{ij} = p_{ji}$. Hence the number of independent stress components reduces to 6. By consideration of the equilibrium of forces on an element of volume, it can be established that

$$p_{ij,i} + F_j = 0 \tag{6}$$

where F_j is the body force per unit volume and a comma denotes differentiation with respect to the variable following it. Since the variation of displacements with respect to time is neglected we are considering static problems in the theory of elasticity.

In (6) we get three equations to determine 6 unknowns. Hence additional relations are necessary. These are provided by the Hooke's generalised stress-strain relations.

$$p_{ij} = C_{ijkl}\, e_{kl} \tag{7}$$

and the strain displacement relations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{8}$$

In (7), it can be established from strain energy considerations that $C_{ijkl} = C_{klij}$, and further by symmetry of stress and

strain components C_{ijkl} is symmetric in i, j as well as in k, l. This gives 21 independent components to C_{ijkl}. In the isotropic case, these reduce to two independent components only and the stresses in terms of strain components can be written as

$$p_{xx} = \lambda(e_{xx}+e_{yy}+e_{zz}) + 2\mu e_{xx}, \quad p_{yy} = \lambda(e_{xx}+e_{yy}+e_{zz}) + 2\mu e_{yy},$$

$$p_{zz} = \lambda(e_{xx}+e_{yy}+e_{zz}) + 2\mu e_{zz}, \quad p_{yz} = 2\mu e_{yz}, \quad p_{zx} = 2\mu e_{zx}, \quad p_{xy} = 2\mu e_{xy}, \tag{9}$$

or in the alternative tensor form these can be written as

$$'p_{ij} = 2\mu\, 'e_{ij}, \qquad p = 3Ke \tag{10}$$

where $'p_{ij} = p_{ij} - \delta_{ij}p/3$, $'e_{ij} = e_{ij} - \delta_{ij}e/3$, $p = p_{ii}$, $e = e_{ii}$ and the constants C_{ijkl} have been suitably replaced by the Lame's constants λ and μ or μ and the bulk modulus K. From equations (9) strains in terms of stresses can be written as

$$e_{xx} = \frac{1}{E}\left\{p_{xx} - \nu(p_{yy}+p_{zz})\right\}, \quad e_{yy} = \frac{1}{E}\left\{p_{yy} - \nu(p_{zz}+p_{xx})\right\}$$

$$e_{zz} = \frac{1}{E}\left\{p_{zz} - \nu(p_{xx}+p_{yy})\right\}, \quad e_{yz} = \frac{p_{yz}}{2\mu}, \quad e_{zx} = \frac{p_{zx}}{2\mu}, \quad e_{xy} = \frac{p_{xy}}{2\mu} \tag{11}$$

where E is the Young's modulus and ν is the Poisson's ratio.

The strain-displacement relations in extended notation

are

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}, \quad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \; e_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right), \; e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{12}$$

Thus by (6), (7) and (12) fifteen equations are obtained in fifteen unknowns p_{ij}, e_{ij}, u_i . The equations (12) satisfy what are usually called the equations of compatibility

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{2\partial^2 e_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{-\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right),$$

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = \frac{2\partial^2 e_{yz}}{\partial z \partial y}, \quad \frac{\partial^2 e_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{-\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right),$$

$$\frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} = \frac{2\partial^2 e_{zx}}{\partial x \partial z}, \quad \frac{\partial^2 e_{zz}}{\partial y \partial x} = \frac{\partial}{\partial z}\left(\frac{-\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right), \tag{13}$$

which physically mean that there are neither overlappings nor voids in the deformed body.

The solution is correct if it satisfies the boundary conditions which are usually given in terms of surface tractions or surface displacements. These conditions generally refer to the whole surface. But in certain circumstances the surface tractions may be prescribed for a part of the boundary and the surface displacements on the remaining portion. Of course, more complicated conditions than these are also known. When surface tractions P_{nx} , P_{ny} , P_{nz}

are prescribed they satisfy the condition

$$P_{nx} = p_{xx}\cos(x,n) + p_{xy}\cos(y,n) + p_{xz}\cos(z,n) ,$$

$$P_{ny} = p_{xy}\cos(x,n) + p_{yy}\cos(y,n) + p_{yz}\cos(z,n) ,$$

$$P_{nz} = p_{xz}\cos(x,n) + p_{yz}\cos(y,n) + p_{zz}\cos(z,n) , \tag{14}$$

where n denotes the direction of the outward drawn normal to the boundary and p_{ij} are the boundary stresses.

In case the surface displacements u_1, u_2, u_3 are prescribed, the displacements u_x, u_y, u_z obtained by solving (6), (9) and (11) should satisfy the conditions

$$u_x = u_1, \quad u_y = u_2, \quad u_z = u_3 . \tag{15}$$

When the boundary conditions refer to boundary tractions only, the stress components should satisfy the equation (6) in addition to the following six equations [6]

$$(1+\nu)\nabla^2 p_{xx} + \frac{\partial^2\theta}{\partial x^2} = 0 , \quad (1+\nu)\nabla^2 p_{yz} + \frac{\partial^2\theta}{\partial y\partial z} = 0$$

$$(1+\nu)\nabla^2 p_{yy} + \frac{\partial^2\theta}{\partial y^2} = 0 , \quad (1+\nu)\nabla^2 p_{zx} + \frac{\partial^2\theta}{\partial z\partial x} = 0$$

$$(1+\nu)\nabla^2 p_{zz} + \frac{\partial^2\theta}{\partial z^2} = 0 , \quad (1+\nu)\nabla^2 p_{xy} + \frac{\partial^2\theta}{\partial x\partial y} = 0 \tag{16}$$

where $\theta = p_{xx} + p_{yy} + p_{zz}$ and body forces are neglected.
Once the stresses are known, strain components are uniquely

determined by Hooke's Law (7) and thus displacements can be evaluated apart from certain constants which account for rigid body displacements, from the relations (8).

If the boundary conditions are in terms of displacements the basic equations reduce to three and these can be put in the form [6] , in terms of cartesian coordinates

$$(\lambda+\mu)\frac{\partial e}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) = 0 ,$$

$$(\lambda+\mu)\frac{\partial e}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) = 0 ,$$

$$(\lambda+\mu)\frac{\partial e}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) = 0 , \tag{17}$$

or in the vector form

$$(\lambda+\mu)\ \mathrm{grad}\ e + \mu\nabla^2(u_x , u_y , u_z) = 0 , \tag{18}$$

or in the alternative form

$$(\lambda+2\mu)\ \mathrm{grad}\ e - 2\mu\ \mathrm{Curl}\ \omega = 0, \tag{19}$$

where $\omega = (\omega_x , \omega_y , \omega_z)$ are components of rotation given by

$$\omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{20}$$

Equations (18) and (19) are particularly useful since gradient, curl and Laplacian operator can be readily transformed to any coordinate system. Once the displacements are known the strains, stresses etc. can easily be determined.

In the case of mixed boundary conditions the choice of the equations is generally suggested by the problem itself.

Thus we have three sets of equations available for us to solve a problem in elasticity theory either the set given by equations (6), (7) and (11), or by (16) or by (17).

It may be remarked that as proved in books on elasticity theory the equations are sufficient and have a unique solution for a simply-connected body. However, for multiply-connected bodies, additional relations are necessary to avoid multi-valued displacements. Not many problems have been worked out in this field and we shall always assume that the displacements are always single-valued.

CHAPTER III

Plane Circular Inclusion

The equations of elasticity theory are given in the previous chapter. Although, in principle, the equations are solvable for any body and under any boundary conditions, in practice they are almost intractable for the general three dimensional problems. A simplification, however, is possible in the case of plane problems. These are of two types (1) plane strain and (2) plane stress.

The plane strain system is usually connected with cylindrical or prismatical bodies in which one dimension, say, parallel to z -axis is very large. Far away from the end sections, the state of stress may be assumed to be the same in all sections perpendicular to the z -axis. Further the displacement of the body in the z -direction is supposed to be negligible. We thus assume that under plane strain conditions u_x, u_y are functions of x, y only and are independent of z, so that

$$u_x = u_x(x,y) \, , \quad u_y = u_y(x,y) \, , \quad u_z = 0 .$$

The equations of equilibrium (6) now reduce to, in cartesian coordinates

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + F_x = 0 \, , \quad \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + F_y = 0 \, . \tag{21}$$

The strain-displacement relations become

$$e_{xx}=\frac{\partial u_x}{\partial x},\quad e_{yy}=\frac{\partial u_y}{\partial y},\quad e_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right),\quad e_{zz}=0,\ e_{xz}=e_{yz}=0. \tag{22}$$

Equations in (13) will reduce to a single equation

$$\frac{\partial^2 e_{yy}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial y^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y}. \tag{23}$$

The stress-strain relations will be

$$p_{xx}=\lambda(e_{xx}+e_{yy})+2\mu e_{xx},\ p_{yy}=\lambda(e_{xx}+e_{yy})+2\mu e_{yy},\ p_{xy}=2\mu e_{xy},$$

$$p_{yz}=p_{zx}=0. \tag{24}$$

It may be noticed that p_{zz} is known once p_{xx} and p_{yy} are known, since the condition

$$e_{zz}=0=\frac{1}{E}\left\{p_{zz}-\nu(p_{xx}+p_{yy})\right\} \quad \text{gives} \quad p_{zz}=\nu(p_{xx}+p_{yy}).$$

This implies that under plane strain system, tractions will have to be applied to balance the stress field p_{zz}. The number of unknowns to be determined in this class of problems reduce to eight p_{ij}, e_{ij}, u_i $(i,j=x,y)$.

In the plane-stress system parallel to x,y plane the stress components $p_{zz}=p_{yz}=p_{zx}=0$. The stress-strain relations will then be

$$p_{xx}=\lambda(e_{xx}+e_{yy}+e_{zz})+2\mu e_{xx},\ p_{yy}=\lambda(e_{xx}+e_{yy}+e_{zz})+2\mu e_{yy},$$

$$p_{xy}=2\mu e_{xy},\ p_{yz}=0,\ p_{zx}=0,$$

$$p_{zz}=0=\lambda(e_{xx}+e_{yy}+e_{zz})+2\mu e_{zz}. \tag{25}$$

The equations of equilibrium are the same as that of the plane-strain system. Further the displacement u_z is known apart from an arbitary constant which accounts for a rigid body displacement if u_x and u_y are known, since

$$\frac{\partial u_z}{\partial x} = -\frac{\partial u_x}{\partial z}, \quad \frac{\partial u_z}{\partial y} = -\frac{\partial u_y}{\partial z}, \quad \frac{\partial u_z}{\partial z} = -\frac{\lambda}{\lambda+2\mu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right),$$

arising from the condition, $e_{xz} = e_{yz} = 0$ and $p_{zz} = 0$. Thus there are effectively eight unknowns of the problem u_i , e_{ij} , p_{ij} which can be obtained from eight equations (11), (21) and (25).

It thus appears that this problem is of the same type as we have in plane strain. But it may carefully be noted that there exists a fundamental difference in the two systems. It is that while the displacements, strain and stress components are independent of the z coordinate in the plane-strain system, in the plane-stress system they are functions of z also. This introduces a great complication in the analysis. The conditions at the boundary also introduce additional complications. We shall not dwell further on this aspect of the problem, as in the rest of the thesis the problem essentially consists of the plane-strain type.

A simplification to deal with plane-stress problems was introduced by Filon [7] in 1903. He introduced the concept of generalised plane stress. This consists in taking $p_{xz} = p_{yz} = 0$ at the surface of the plates which essentially account for plane stress problems and $p_{zz} = 0$ throughout the plate. Instead

of considering the stress, strain and displacement components, he considered their average values. Thus if $\bar{p}_{xx}, \ldots, \ldots, \bar{e}_{xx}, \ldots, \ldots, \bar{u}_x, \ldots, \ldots,$ are the average values,

$$\bar{p}_{xx} = \frac{1}{2h}\int_{-h}^{h} p_{xx}\,dz, \ldots, \ldots, \quad \bar{e}_{xx} = \frac{1}{2h}\int_{-h}^{h} e_{xx}\,dz, \ldots, \ldots,$$

$$\bar{u}_x = \frac{1}{2h}\int_{-h}^{h} u_x\,dz, \ldots, \ldots .$$

Since the plate is thin, it is reasonable to assume that the average values cannot differ substantially from the true values. The condition $p_{zz} = 0$ gives, in terms of the average values,

$$\bar{e}_{zz} = -\frac{\lambda}{\lambda+2\mu}\left(\bar{e}_{xx} + \bar{e}_{yy}\right) . \tag{26}$$

Substituting these values of strains in the stress-strain relations (25)

$$\bar{p}_{xx} = \frac{2\lambda\mu}{\lambda+2\mu}\left(\bar{e}_{xx} + \bar{e}_{yy}\right) + 2\mu\bar{e}_{xx}, \quad \bar{p}_{yy} = \frac{2\lambda\mu}{\lambda+2\mu}\left(\bar{e}_{xx} + \bar{e}_{yy}\right) + 2\mu\bar{e}_{yy} . \tag{27}$$

We see that these relations are the same as (24) except that λ has been replaced by λ^* where

$$\lambda^* = \frac{2\lambda\mu}{\lambda+2\mu} . \tag{28}$$

Here each of the components of average stresses and strains will be a function of (x, y) only.

Consider now the case of a plane circular inclusion, of radius a tending to attain a size $a(1+\delta)$ in the absence of the matrix. What will be the equilibrium size and shape of the inclusion, and the accompanying elastic field because of the presence of the matrix ?

Following again the method of Nabarro, we first calculate the displacement of the inclusion from its free boundary due to a pressure P under plane strain.

Let us suppose that in the equilibrium position the radius of the equilibrium boundary is $a(1+\epsilon)$. The resulting displacement due to the pressure P in the inclusion at the boundary is

$$U_r = a(\epsilon - \delta), \quad U_\theta = 0, \quad U_z = 0. \tag{29}$$

For the inclusion it is readily proved that

$$P_{rr} = P_{\theta\theta} = -P, \quad P_{zz} = -2\nu P. \tag{30}$$

Therefore,

$$\mathcal{E}_{\theta\theta} = \frac{U_r}{r} = -\frac{P(1+\nu)(1-2\nu)}{E},$$

hence

$$U_r = -\frac{Pr}{2(\lambda+\mu)} \tag{31}$$

Equating the boundary displacement given in equation (29) to that determined by equation (31), we get

$$\epsilon - \delta = -\frac{P}{2(\lambda+\mu)} \cdot \tag{32}$$

For the matrix, again from the equilibrium considerations, the internal pressure is the same P . Whence the boundary conditions are at $r = a$, $p_{rr} = -P$, $u_r = a\epsilon$.

The solution of an infinite region with a circular hole is well known [8] . This gives the stress field

$$p_{rr} = -\frac{Pa^2}{r^2}, \quad p_{\theta\theta} = \frac{Pa^2}{r^2}, \quad p_{zz} = 0, \quad p_{r\theta} = p_{rz} = p_{\theta z} = 0, \tag{33}$$

whence

$$e_{\theta\theta} = \frac{Pa^2}{2\mu r^2}, \tag{34}$$

and therefore

$$u_r = \frac{Pa^2}{2\mu r} \cdot \tag{35}$$

Equating this to the boundary displacement $a\epsilon$ at $r = a$, we get

$$\epsilon = \frac{P}{2\mu} \cdot \tag{36}$$

From equations (32) and (36)

$$\epsilon = \frac{(\lambda+\mu)}{(\lambda+2\mu)}\,\delta\,, \quad P = \frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)}\,\delta\,. \tag{37}$$

The strain energy in the inclusion and the matrix will respectively be

$$W_I = \frac{2(\lambda+\mu)\delta^2\mu^2}{(\lambda+2\mu)^2}\,\pi a^2\,,$$

$$W_m = \frac{2\mu(\lambda+\mu)^2\delta^2}{(\lambda+2\mu)^2}\,\pi a^2. \tag{38}$$

As we shall be concerned with the expressions for strain energy, its connection with the work done at the surface and some basic theorems connected with it to find the solution of the problems, we state these briefly in the next chapter.

CHAPTER IV

Strain Energy

Consider an unstressed uniform elastic bar with its axis along x -axis. Let a gradually increasing load be applied to the bar in this direction. At any instant when the body is in elastic equlibrium, the only non-zero stress component will be p_{xx} along x -direction. As the force is being applied from initial unstressed to stressed configuration the average force acting on an elemental volume $dx\,dy\,dz$ can be taken to be $p_{xx}\,dy\,dz/2$. As a result of the stress the particles constituting the body are relatively displaced. If e_{xx} be the strain-component parallel to x -axis, the displacement parallel to this direction is $e_{xx}\,dx$. The work done by the force $p_{xx}\,dy\,dz/2$ is, therefore,

$$dW = \frac{1}{2}\, p_{xx} e_{xx}\, dx\, dy\, dz.$$

By the first law of thermodynamics the work done on a body is stored in the form of potential energy and heat energy. It may be proved that under reversible adiabatic or isothermal conditions the change in the heat energy is stored in the form of strain energy.

In a three dimensional case when a body is loaded, all the stress components will be present. By conservation of energy it can be seen that under conditions of gradual loading,

the strain energy in an element of volume will be

$$dw = \frac{1}{2}\left(p_{xx}e_{xx} + p_{yy}e_{yy} + p_{zz}e_{zz} + 2p_{yz}e_{yz} + 2p_{zx}e_{zx} + 2p_{xy}e_{xy}\right) dx\,dy\,dz \ . \quad (39)$$

Therefore, the strain energy density per unit volume is

$$W_0 = \frac{1}{2}\left(p_{xx}e_{xx} + p_{yy}e_{yy} + p_{zz}e_{zz} + 2p_{yz}e_{yz} + 2p_{zx}e_{zx} + 2p_{xy}e_{xy}\right),$$
$$= \frac{1}{2} p_{ij}e_{ij} \ . \quad (40)$$

Since the energy is a scalar quantity, the energy stored in the body may be taken as the scalar sum of the energy stored in all its elements. For a continuously distributed matter the total strain energy may be obtained by integrating dw in equation (39) throughout the volume.

By making use of the stress-strain relations (7) for an elastic body the strain energy density may be put in terms of stress or strain components only. For isotropic bodies, expressing the strain components in (40) in terms of stresses by equations (11), the strain energy per unit volume is given by

$$W_0 = \frac{1}{2E}\left(p_{xx}^2 + p_{yy}^2 + p_{zz}^2\right) - \frac{\nu}{E}\left(p_{xx}p_{yy} + p_{yy}p_{zz} + p_{zz}p_{xx}\right) + \frac{1}{2\mu}\left(p_{xy}^2 + p_{yz}^2 + p_{zx}^2\right). \quad (41)$$

Similarly expressing stress components in terms of strains by

equations (9), equation (40) can be written as

$$W_o = \frac{1}{2}\lambda(e_{xx}+e_{yy}+e_{zz})^2+\mu(e_{xx}^2+e_{yy}^2+e_{zz}^2)+2\mu(e_{xy}^2+e_{yz}^2+e_{zx}^2). \tag{42}$$

These relations may be put in the form

$$W_o = \frac{3(1-2\nu)}{2E}\left(\frac{p_{xx}+p_{yy}+p_{zz}}{3}\right)^2+\frac{(1+\nu)}{6E}\left\{(p_{xx}-p_{yy})^2+(p_{yy}-p_{zz})^2+(p_{zz}-p_{zx})^2+ 6(p_{xy}^2+p_{yz}^2+p_{zx}^2)\right\}, \tag{43}$$

or in the alternative form

$$W_o = \frac{E}{6(1-2\nu)}(e_{xx}+e_{yy}+e_{zz})^2+\frac{E}{6(1+\nu)}\left\{(e_{xx}-e_{yy})^2+(e_{yy}-e_{zz})^2+(e_{zz}-e_{xx})^2 +6(e_{xy}^2+e_{yz}^2+e_{zx}^2)\right\}. \tag{44}$$

In terms of principal stresses p_{11}, p_{22}, p_{33} or strains e_{11}, e_{22}, e_{33} these equations reduce to

$$W_o = \frac{3(1-2\nu)}{E}\left(\frac{p_{11}+p_{22}+p_{33}}{3}\right)^2+\frac{1+\nu}{6E}\left\{(p_{11}-p_{22})^2+(p_{22}-p_{33})^2+(p_{33}-p_{11})^2\right\}, \tag{45}$$

or

$$W_o = \frac{E}{6(1-2\nu)}(e_{11}+e_{22}+e_{33})^2+\frac{E}{6(1+\nu)}\left\{(e_{11}-e_{22})^2+(e_{22}-e_{33})^2+(e_{33}-e_{11})^2\right\}. \tag{46}$$

The terms in the brackets on the right hand side of (43) and (44) or (45) and (46) can be physically interpreted. In the first brackets on the right hand side of equations (43) and

(44), $(p_{xx} + p_{yy} + p_{zz})/3$ and $e_{xx} + e_{yy} + e_{zz}$ may be taken as the average normal stress and dilation respectively. Similar remarks apply to equations (45) and (46) where we have taken the principal stresses or strains. These account for the change in volume of element. The terms in the second bracket in each of the equations (43) to (46), pertain to the distortion. This distinction is sometimes necessary, as in plasticity theory or soil mechanics where the mechanical behaviour of a material depends upon those combinations of stress or strain components which change the shape of the body.

This strain energy is related to the work done by the external forces by Clapeyron's theorem [8]. This theorem states " If a body is in equilibrium under a given system of body forces F_i and surface forces T_i then the strain energy of deformation is equal to one-half the work that would be done by the external forces (of the equilibrium state) acting through the displacements u_i from the unstressed state to the state of equilibrium. " This theorem can be easily proved as follows. Work done by the external forces is given by

$$\int_V F_i u_i \, dv + \int_S T_i u_i \, ds = \int_V F_i u_i \, dv + \int_S p_{ij} u_i n_j \, ds$$

$$= \int_V F_i u_i \, dv + \int_V (p_{ij} u_i)_{,j} \, dv$$

$$= \int_V u_j (p_{ij,i} + F_j) \, dv + \frac{1}{2} \int_V p_{ij} (u_{i,j} + u_{j,i}) \, dv$$

$$= \int_v u_i (p_{ij,j} + F_i) dv + \int_v p_{ij} e_{ij} dv$$

since $p_{ij} n_j = T_i$, $e_{ij} = (u_{i,j} + u_{j,i})/2$ and $u_{i,j} = (u_{i,j} + u_{j,i})/2$. Note that in the above derivation there is symmetry in i and j . Here n_j are the components of the outward drawn normal n at the surface. Because of stress equilibrium equations (6), we get

$$\int p_{ij} e_{ij} dV = 2 \int W_0 dV , \tag{47}$$

by making use of equation (40).

Strain energy methods can be utilised through the minimum energy theorem. This theorem states [9] "The displacement which satisfies the differential equations of equilibrium, as well as the conditions at the bounding surface, yields a smaller value for the potential energy of deformation than any other displacement, which satisfies the same conditions at the bounding surface." Here a distinction is made between the potential energy and strain energy.

Consider a body whose boundary conditions are given only in terms of surface displacements at each point of the bounding surface. Let the correct displacement function which satisfies the strain displacement relations, the compatibility relations, stress-strain relations, equations of equilibrium and boundary conditions be u_i . Let $u_i + \delta u_i$ be any other displacement which satisfies only the boundary conditions. Obviously $\delta u_i = 0$ at the boundary. It may be emphasised that we are considering that case only where the boundary conditions consist of

displacements only. Let e_{ij} and δe_{ij} be the strain components corresponding to u_i and δu_i .

The work done by the external forces F_i and T_i in a displacement δu_i is given by

$$\delta W = \int F_i \, \delta u_i \, dv , \tag{48}$$

as no work is done by the surface force T_i (δu_i being equal to zero on the surface), whence

$$\delta \left(W - \int_V F_i u_i \, dv \right) = 0 , \tag{49}$$

since $\delta \int F_i u_i \, dv = \int F_i \, \delta u_i \, dv$ as F_i and dv remain constant for an arbitrary variation of u_i . Further since $W = \int W_0 \, dv$, therefore by (49)

$$\delta \left[\int_V W_0 \, dv - \int_V F_i u_i \, dv \right] = 0 \tag{50}$$

The expression $\int W_0 \, dv - \int F_i u_i \, dv$ is termed as potential energy and this has an extremum value by equation (50). We can further prove that this potential energy is minimum. Let Δu_i be any arbitrary increment to u_i . The increment in the strain energy due to displacement changing from u_i to $u_i + \Delta u_i$ can be determined by first finding ΔW_0 which is given by

$$\Delta W_0 = \left\{ \frac{\lambda}{2} (e_{ij} \delta_{ij})^2 + \mu e_{ij} e_{ij} \right\}_{u_i + \Delta u_i} - \left\{ \frac{\lambda}{2} (e_{ij} \delta_{ij})^2 + \mu e_{ij} e_{ij} \right\}_{u_i} .$$

Simplifying this and remembering that $e_{ij} = (u_{i,j} + u_{j,i}) / 2$ we get

$$\begin{aligned} \Delta W_0 &= \lambda e_{kl} \delta_{kl} (\Delta u_i)_{,j} + 2 \mu e_{ij} (\Delta u_i)_{,j} + f(\Delta e_{ij}) \\ &= p_{ij} (\Delta u_i)_{,j} + f(\Delta e_{ij}) \end{aligned} \tag{51}$$

The total change of the energy in the body is then

$$\begin{aligned}
\Delta W = \int \Delta W_0 \, dv &= \int p_{ij} (\Delta u_i)_{,j} \, dv + \int f(\Delta e_{ij}) \, dv , \\
&= \int (p_{ij} \Delta u_i)_{,j} \, dv - \int \Delta u_i \, p_{ij,j} \, dv + \int f(\Delta e_{ij}) \, dv , \\
&= \int p_{ij} \Delta u_i n_j \, ds - \int p_{ij,j} \Delta u_i \, dv + \int f(\Delta e_{ij}) \, dv , \\
&= \Delta \int F_i u_i \, dv + \int f(\Delta e_{ij}) \, dv ,
\end{aligned} \tag{52}$$

where use has been made of equilibrium equations and further of the fact that Δu_i vanishes at the boundary surface. Also Δ can be brought out of the sign of integration by the same arguments which have been used for $\int F_i \, \delta u_i \, dv$. Therefore, the increment in the potential energy is given by

$$\Delta \left(\int W_0 \, dv - \int F_i u_i \, dv \right) = \int f(\Delta e_{ij}) \, dv . \tag{53}$$

The function $f(\Delta e_{ij})$ is a positive definite quadratic function and therefore the increment due to an arbitrary displacement Δu_i is positive. Therefore, the potential energy derived from the displacement which satisfies the equations of equilibrium and boundary condition is a minimum.

In the subsequent analysis we shall be neglecting the body forces. The potential energy and strain energy are then identical and we obtain from equation (53)

$$\Delta W = \Delta \int W_0 \, dv = \int f(\Delta e_{ij}) \, dv \tag{54}$$

which proves that the elastic strain energy is a minimum in equilibrium position.

CHAPTER V

Circular Inclusion - Strain Energy Method

In previous chapter, expressions for strain energy in terms of stresses and or strains have been found for a three-dimensional case. In plane problems, these reduce to simpler expressions. For example, in the case of plane stress, since $p_{zz} = p_{zx} = p_{yz} = 0$ the strain energy density given by equation (41) can be written as

$$W_0 = \frac{1-\nu}{4E}(p_{xx}+p_{yy})^2 + \frac{1+\nu}{4E}\left\{(p_{yy}-p_{xx})^2 + 4p_{xy}^2\right\} \tag{55}$$

in terms of stress components. Similarly for plane strain case since $p_{zz} = \nu(p_{xx}+p_{yy})$, $p_{yz} = 0$, $p_{zx} = 0$ it becomes,

$$W_0 = \frac{1+\nu}{4E}\left[(1-2\nu)(p_{xx}+p_{yy})^2 + \left\{(p_{yy}-p_{xx})^2 + 4p_{xy}^2\right\}\right] \tag{56}$$

It is known that the expressions $(p_{xx}+p_{yy})^2$ and $\left\{(p_{yy}-p_{xx})^2+4p_{xy}^2\right\}$ are invariants for transformation of coordinate system, and hence are applicable even when the cartesian system changes to any curvilinear coordinate system. It may however be remarked that the invariance of the expression for strain energy is otherwise also obvious, since it is a physical entity and would not depend upon the coordinate system. In the next chapter we shall observe that simple complex variable expressions are available for these invariants.

The theorem that the strain energy is minimum in elastic

equilibrium position when the body forces are zero and the boundary conditions refer to surface displacement only, remains true. This theorem has been applied to plane strain inclusion problems by the following procedure.

In order to apply this theorem the strain energy in the system of inclusion and matrix is to be calculated. To do this the two sub-systems consisting of inclusion and matrix are dealt separately. We imagine that the inclusion has occupied the free-surface. We consider a possible equilibrium interface. At this stage suitable unknown parameter or parameters are introduced. Thus the elastic boundary displacements are known in terms of these parameters. Such a procedure is similar to St.Venant's semi-inverse methods which he applied to torsion problems etc. in elasticity. Solution of the problem when boundary displacements are known, is obtained in a relatively simple manner, specially for finite simply-connected bodies. It may, however, be remarked that such methods are amenable to analytical solutions only when the figure can be mapped on to a unit circle by simple expressions. For more complicated shapes recourse has to be taken to numerical procedures. Knowing the stress-strain field by the solution of the fundamental problem, the strain energy in the inclusion can be calculated by equation (56) if we are dealing with isotropic plane strain case.

The calculation of the strain energy in the matrix again depends on solving a second fundamental problem for it. Because the displacements on the boundary are known in terms of the

parameter introduced for the elastic displacement of inclusion. For the case of an infinite region bounded internally by a circle the form of the solution is known [6] . The determination of the strain energy density for this is relatively not complicated and can be obtained by equation (56). But for other shapes, the solution of the fundamental problem will be so complicated that the strain energy cannot be easily determined. In such cases one has to use the Clapeyron's theorem.

The strain energy per unit thickness in both of them is obtained by integrating the expressions over their area. The total strain energy per unit thickness, therefore, in the medium is obtained by the scalar addition of the two. This total strain energy is then minimised with respect to the assumed displacement parameters. This gives a set of equations to determine the parameters which in turn gives the correct equilibrium boundary.

The determination of the stresses and strains every where is then just a matter of putting the correct value of displacement parameters. As already remarked the results of any elastic problem are correct if they satisfy the boundary conditions, equilibrium equations and compatibility relations. The boundary conditions in this case are: there should be continuity of displacements and also of the normal and shear stresses at the equilibrium boundary of the two systems. Continuity of displacements, equations of equilibrium and compatibility are already taken into account in the methods we

follow. The results should also satisfy the continuity of normal and shear stresses, which have to be substantiated in each case.

In subsequent work the elastic properties of the inclusion have been taken to be different from those of the matrix. This will provide us complete flexibility in discussing a wide range of isotropic inhomogeneity problems in any material. Thus if the Lamé's constant λ_1 of the inclusion material tends to infinity and μ_1 remaining finite, it would be incompressible but still would permit a change in its shape. On the other hand if both the Lamé's constants λ_1 and μ_1 tend to infinity, the inclusion is rigid permitting a change neither in volume nor in shape. On the other extreme, we could consider the case when the Lame's constants of the inclusion tend to zero. This would account for a cavity in the material. If λ_1, μ_1 tend to λ, μ -the Lame's constants of the matrix material, then the inclusion will be of the same material as that of the matrix. The mechanical properties of the inclusion will henceforth be identified by suffix 1.

As a simple application, we consider here the case of a circular inclusion already dealt with in chapter 3. Also, the same notations for the initial boundary, free surface of the inclusion, the equilibrium boundary, the stresses and the strains etc. are used.

The elastic displacement of the boundary of the inclusion is reckoned from free surface to that of the equilibrium boundary which is assumed to be a concentric circle of radius

$a(1+\epsilon)$. This displacement is equal to $a(\epsilon-\delta)$. Given the boundary displacement, the strains, stresses, etc., are found by the application of elasticity theory cited in chapters 2 and 3. However, in this case, the solution is known. If U_r, U_θ are the displacement components at a distance r from the centre

$$U_r = (\epsilon - \delta) r, \qquad U_\theta = 0$$

whence the radial, hoop and shear strains are

$$\mathcal{E}_{rr} = \mathcal{E}_{\theta\theta} = (\epsilon - \delta), \qquad \mathcal{E}_{r\theta} = 0$$

and therefore, the dilation field is

$$\Delta = 2(\epsilon - \delta).$$

The corresponding stress field can be obtained by using the Hooke's Law, and is

$$P_{rr} = P_{\theta\theta} = 2(\lambda_1 + \mu_1)(\epsilon - \delta), \quad P_{r\theta} = 0. \tag{57}$$

The strain energy density by equation (56) (suitable modification for polar coordinates can be made) is $2(\lambda_1+\mu_1)(\epsilon-\delta)^2$. Therefore the elastic strain energy in the inclusion (assuming the thickness to be unity) is

$$W_I = 2(\lambda_1 + \mu_1)(\epsilon - \delta)^2 \pi a^2. \tag{58}$$

The matrix undergoes a radial displacement of $a\epsilon$ at the boundary. For a uniform normal pressure applied at the cylindrical cavity in an infinite medium, the stresses are of

the form [6]

$$p_{rr} = \frac{A}{r^2}\ , \quad p_{\theta\theta} = -\frac{A}{r^2}\ , \quad p_{r\theta} = 0\ . \tag{59}$$

The corresponding strain components by Hooke's Law are

$$e_{rr} = \frac{A}{2\mu r^2}\ , \quad e_{\theta\theta} = \frac{-A}{2\mu r^2}\ , \quad e_{r\theta} = 0\ , \tag{60}$$

and the displacement components are

$$u_r = -\frac{A}{2\mu r}\ , \qquad u_\theta = 0\ . \tag{61}$$

Equating the value of u_r at $r = a$ to the value $a\epsilon$ we obtain

$$A = -2\mu\epsilon a^2.$$

Substitution of this value of A in equations (59),(60) and (61) gives the stresses, strains and displacement at a distance r . Therefore we get

$$p_{rr} = -\frac{2\mu\epsilon a^2}{r^2}\ , \quad p_{\theta\theta} = \frac{2\mu\epsilon a^2}{r^2}\ , \quad p_{r\theta} = 0\ ,$$

$$e_{rr} = -\frac{\epsilon a^2}{r^2}\ , \quad e_{\theta\theta} = \frac{\epsilon a^2}{r^2}\ , \quad e_{r\theta} = 0\ . \tag{62}$$

The dilation is $\Delta = 0$. The strain energy density, by equation (56), in the matrix is $2\mu\epsilon^2 a^4/r^4$. The strain energy in a ring of width dr is $2\mu\epsilon^2 a^4 2\pi r\, dr/r^4$. Hence the total strain energy in the matrix is

$$W_m = \int_0^\infty 4\mu\epsilon^2 a^4 \pi \frac{dr}{r^3} = 2\mu\epsilon^2\pi a^2. \tag{63}$$

The total strain energy of the system is

$$W = W_I + W_m = 2\pi a^2 \left\{ (\lambda_1 + \mu_1)\,(\epsilon - \delta)^2 + \mu \epsilon^2 \right\}. \tag{64}$$

Minimising W with respect to ϵ by proving the second derivative $d^2W/d\epsilon^2$ to be positive for the value of ϵ obtained by putting $dW/d\epsilon$ equal to zero, we get the equilibrium value of ϵ which is given by

$$\epsilon = \frac{(\lambda_1 + \mu_1)}{(\lambda_1 + \mu_1 + \mu)}\,\delta. \tag{65}$$

The continuity of the normal and shear stresses at the interface of the inclusion and matrix can be easily verified to substantiate the above result. By putting $r = a$ and the value of ϵ from equation (65) in equations (57) and (62) and using a superscript b to indicate the equilibrium boundary value, we get

$$P^b_{rr} = P^b_{\theta\theta} = \frac{-2(\lambda_1 + \mu_1)\mu}{\lambda_1 + \mu_1 + \mu}\,\delta, \qquad P_{r\theta} = 0,$$

$$p^b_{rr} = -\frac{2(\lambda_1 + \mu_1)\mu}{(\lambda_1 + \mu_1 + \mu)}\,\delta, \quad p^b_{\theta\theta} = \frac{2(\lambda_1 + \mu_1)\mu\delta}{(\lambda_1 + \mu_1 + \mu)}, \quad p^b_{r\theta} = 0. \tag{66}$$

It is clear from equations (66) that the normal and shear stresses are continuous and the jump in the hoop stress is given by

$$p^b_{\theta\theta} - P^b_{\theta\theta} = \frac{4(\lambda_1 + \mu_1)\mu}{(\lambda_1 + \mu_1 + \mu)}\,\delta. \tag{67}$$

Substituting the value of ϵ in equations (58) and (63) the strain energy in the inclusion and matrix are respectively

$$W_I = \frac{2(\lambda_1+\mu_1)\,\mu^2}{(\lambda_1+\mu_1+\mu)^2}\,\pi a^2\delta^2, \qquad W_m = \frac{2\mu\,(\lambda_1+\mu_1)^2}{(\lambda_1+\mu_1+\mu)^2}\,\pi a^2\delta^2.$$

Equation (65) for the value of ϵ can be written, in terms of the Young's moduli and Poisson's ratios of the materials involved as

$$\epsilon = \frac{E_1(1+\nu)\,\delta}{E_1(1+\nu)+E(1+\nu_1)(1-2\nu_1)} \,. \tag{68}$$

By putting λ_1 and μ_1 equal to λ, μ in equation (65) or $\nu = \nu_1$ and $E = E_1$ in equation (68), the expression for ϵ reduces to the known value as given by equation (37). This is a case when the inclusion and matrix are of the same material. As another particular case we might take the inclusion material to be incompressible i.e., Poisson's ratio $\nu_1 = \frac{1}{2}$. Then we get $\epsilon = \delta$, as it should. The result is the same even if E_1 tends to ∞ i.e., inclusion material is rigid. On the other hand if the matrix is rigid λ, μ tend to infinity then $\epsilon = 0$ which is correct on physical grounds also. But if the matrix is only incompressible then

$$\epsilon = \frac{3E_1\,\delta}{3E_1+2E(1+\nu_1)(1-2\nu_1)} \,.$$

The solution to spherical inclusion problem by a similar method is given in appendix I.

CHAPTER VI

Complex Variable Method

The basic equations of elasticity theory for the plane case have been given in chapter III. In the absence of the body forces the equations of equilibrium (21), can be written as

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} = 0 , \quad \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} = 0 . \tag{69}$$

In plane problems the compatibility equation (23) can be written in terms of stress components only, by making use of stress-strain relations and the two equations of equilibrium (69). After a little simplification, these reduce to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (p_{xx} + p_{yy}) = 0 , \tag{70}$$

which shows that the stress $p_{xx} + p_{yy}$ satisfies the Laplace's equation.

As already remarked the boundary conditions can be specified in terms of the boundary tractions or boundary displacements. These are usually referred to as first fundamental problem or the second fundamental problem respectively. Many variations of the boundary conditions are available in literature and occur in practice. For example, one might consider the mixed problem, i.e., when on a part of the boundary, surface tractions are prescribed and on the remaining part, the displacements are prescribed. As another example, at

each point of the surface, one component of surface traction and the other component of displacement or vice-versa, may be given. Finally at each point, one might consider a relationship between the surface traction and surface displacement. However, as we shall be primarly concerned with the problems of the type of first and second fundamental problems, we shall consider them in a little detail.

In the case of the first fundamental problem for plane problems equations (14) reduce to

$$P_{nx} = p_{xx}\cos(x,n) + p_{xy}\cos(y,n), \quad P_{ny} = p_{xy}\cos(x,n) + p_{yy}\cos(y,n). \tag{71}$$

Similarly, for the second fundamental problem equations (15) become

$$u_x = u_1(s) \quad , \quad u_y = u_2(s), \tag{72}$$

where $u_1(s)$ and $u_2(s)$ are the displacements given as a function of the parameter s usually the arc length.

The solutions of these equations (69), (70) and (71) or (72) are generally obtained by introducing a stress function χ first introduced by the English Astronomer Airy in 1862, such that

$$p_{xx} = \frac{\partial^2 \chi}{\partial y^2} \quad , \quad p_{yy} = \frac{\partial^2 \chi}{\partial x^2} \quad , \quad p_{xy} = \frac{-\partial^2 \chi}{\partial x \partial y}. \tag{73}$$

It is evident that equations (73) satisfy the equilibrium equations (69). The substitution of equations (73) in the stress compatibility equation (70) yields the biharmonic equation

$$\nabla^4 \chi = \frac{\partial^4 \chi}{\partial x^4} + \frac{2\partial^4 \chi}{\partial x^2 \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} = 0 . \tag{74}$$

This biharmonic equation is to be solved such that it satisfies the appropriate conditions given at the surface in terms of tractions or displacements.

A direct method based on complex variable techniques of solving plane problems was first suggested by A.C. stevenson [10] in U.K. and Kolosov [11] in U.S.S.R. The results are available in the books by Mushkhilishvli [12] Green and Zerna [13] Sokolnikoff [8] and others.

The method is based on finding two complex functions $\phi(z)$ and $\psi(z)$ where $z = (x + iy)$ is a complex variable. It can be verified that equation (74) in terms of z and its conjugate $\bar{z}$ can be written as

$$\frac{\partial^4 \chi}{\partial z^2 \partial \bar{z}^2} = 0 . \tag{75}$$

Integrating this equation one gets

$$\chi = \bar{z} f(z) + g(z) + z\overline{h(z)} + \overline{k(z)}. \tag{76}$$

Taking the conjugate of equation (76) and noting that χ is real,

$$\chi = z\overline{f(z)} + \overline{g(z)} + \bar{z}\, h(z) + k(z) . \tag{77}$$

Adding, equations (76) and (77),

$$2\chi = z\overline{(f+h)} + \bar{z}\,(f+h) + (g+k) + \overline{(g+k)} .$$

or

$$\chi = \frac{1}{2}\left\{ z\,\overline{\phi(z)} + \bar{z}\,\phi(z) + \theta(z) + \overline{\theta(z)} \right\}. \tag{78}$$

It can be readily proved by making use of equations (73) and (78) that the stress components are given by

$$p_{xx} + p_{yy} = 4\,\mathrm{Re}\,\phi'(z),\quad p_{yy} - p_{xx} + 2i p_{xy} = 2\left\{ \bar{z}\,\phi''(z) + \psi'(z) \right\}, \tag{79}$$

in which $\theta(z)$ has been replaced by the relation $\theta'(z) = \psi(z)$. Making use of the stress-strain relations in plane strain problems and equations(79) the strain components can be obtained and are given by

$$e_{xx} + e_{yy} = \frac{4\,\mathrm{Re}\,\phi'(z)}{(\lambda + 2\mu)},\quad e_{yy} - e_{xx} + 2i e_{xy} = \frac{1}{\mu}\left\{ \bar{z}\,\phi''(z) + \psi'(z) \right\}. \tag{80}$$

The boundary conditions expressed by equations (71) can be written in terms of the Airy's stress function, as

$$p_{nx} = p_{xx}\frac{dy}{ds} - p_{xy}\frac{dx}{ds} = \frac{\partial^2 \chi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 \chi}{\partial x \partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial \chi}{\partial y}\right),$$

$$p_{ny} = -\frac{d}{ds}\left(\frac{\partial \chi}{\partial x}\right). \tag{81}$$

Integrating these equations and making use of equation (78), it can be proved that

$$\int d\left\{ \frac{\partial \chi}{\partial x} + i\,\frac{\partial \chi}{\partial y} \right\} = \phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)} = f_1 + i f_2 + C, \tag{82}$$

where C is a constant.

From equations (24) one can obtain strains in terms of stress

components in plane-strain problems as

$$e_{xx} = \frac{1}{2\mu}\left\{-p_{yy} + (1-\nu)(p_{xx}+p_{yy})\right\}, \quad e_{yy} = \left\{-p_{xx} + (1-\nu)(p_{xx}+p_{yy})\right\}\frac{1}{2\mu},$$
$$e_{xy} = \frac{1}{2\mu}\, p_{xy}. \tag{83}$$

Substituting for $p_{xx} + p_{yy}$ by equations (79), making use of equations (73) and integrating equations (83), and remembering that $e_{xx} = \partial u_x/\partial x$, $e_{yy} = \partial u_y/\partial y$, $e_{xy} = \frac{1}{2}(\partial u_y/\partial x + \partial u_x/\partial y)$, we obtain

$$2\mu u_x = \frac{-\partial \chi}{\partial x} + (1-\nu)\operatorname{Re}\phi(z),$$
$$2\mu u_y = \frac{-\partial \chi}{\partial y} + (1-\nu)\operatorname{Re}\phi(z). \tag{84}$$

From equations (84) and (78) and remembering that $\theta'(z) = \psi(z)$, it is easily seen that

$$2\mu(u_x + i u_y) = k\,\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi(z)}, \tag{85}$$

where $k = 3-4\nu$ in the plane strain problems. The constants of integration expressing rigid body motion are neglected. As already remarked the solution for the generalised plane stress case can be obtained by replacing λ by λ^* given by equation (28). Whence in plane-stress case k is replaced by k^* which is given by $k^* = (3-\nu)/(1+\nu)$. If the boundary conditions are in terms of the displacements then equations (72) in terms of the complex function, by (85), will become

$$2\mu\left(u_1(s) + i u_2(s)\right) = k\,\phi(t) - t\,\overline{\phi'(t)} - \overline{\psi(t)} \tag{86}$$

where t is the boundary value of z.

The boundary value problems of simply connected regions, finite or infinite, are therefore solved if the two complex functions ϕ and ψ are determined. Conformal mapping and Cauchy integrals are effective tools for this. As we shall be concerned mainly with the second fundamental problem, the method of finding ϕ and ψ for such a case is explained.

The analytic function $z = \omega(\zeta)$ is used to map the region in the z-plane into the unit circle $|\zeta| \leq 1$ in the ζ -plane. The boundary condition (86) can now be written by putting $\phi(\omega(\zeta)) = F(\zeta)$, $\psi(\omega(\zeta)) = G(\zeta)$ and $\zeta = \sigma = e^{i\theta}$ boundary value of ζ, as

$$k F(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{F'(\sigma)} - \overline{G(\sigma)} = 2\mu \{u_1(\sigma) + i u_2(\sigma)\} \equiv 2\mu u(\sigma). \tag{87}$$

Therefore taking the conjugate of the above

$$k \overline{F(\sigma)} - \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} F'(\sigma) - G(\sigma) = 2\mu \overline{u(\sigma)}. \tag{88}$$

Equation (87) can be reduced to an integro-differential equation. Multiplying both sides of the equation by $d\sigma / 2\pi i(\sigma - \zeta)$ and integrating over the contour of the unit circle, we get

$$\frac{k}{2\pi i}\int_\sigma \frac{F(\sigma)\, d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int_\sigma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{F'(\sigma)}}{\sigma - \zeta} d\sigma - \frac{1}{2\pi i}\int_\sigma \frac{\overline{G(\sigma)}}{\sigma - \zeta} d\sigma = \frac{2\mu}{2\pi i}\int_\sigma \frac{u(\sigma)}{\sigma - \zeta} d\sigma. \tag{89}$$

Now, noting that if $H(\zeta)$ is analytic within and on $|\zeta| \leq 1$,

$$\frac{1}{2\pi i}\int_\sigma \frac{H(\sigma)}{\sigma - \zeta} d\sigma = H(\zeta), \qquad \frac{1}{2\pi i}\int_\sigma \frac{\overline{H(\sigma)}}{\sigma - \zeta} d\sigma = \overline{H(0)}. \tag{90}$$

Therefore equation (89) becomes,

$$k F(\zeta) - \frac{1}{2\pi i}\int \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{F'(\sigma)}}{\sigma-\zeta} d\sigma - G(0) = 2\mu u(\zeta) \tag{91}$$

By solving this equation, $F(\zeta)$ can be obtained. The constant $G(0)$ in this equation may be evaluated by imposing the condition $F(0) = 0$. When $F(\zeta)$ is known $G(\zeta)$ may be determined by equation (88) and Cauchy's integral formula (90).

Determination of the complex functions $F(\zeta)$ and $G(\zeta)$ will immediately enable us to find $\phi(z)$ and $\psi(z)$ by inverse transformation from ζ-plane to z plane. After knowing ϕ and ψ it is possible to know the cartesian stress components by equation (77). The stress components in any curvilinear (ξ, η) system can be obtained from the relations

$$P_{xx} + P_{yy} = P_{\xi\xi} + P_{\eta\eta}, \quad P_{yy} - P_{xx} + 2i P_{xy} = (P_{\eta\eta} - P_{\xi\xi} + 2i P_{\xi\eta}) e^{-2i\theta}, \tag{92}$$

where θ is the angle between the tangent to the curve $\eta =$ constant, in the direction ξ increasing, and the x-axis. It can be seen from equations (79) and (92) that the quantities $4\,\mathrm{Re}\,\phi'(z)$ and $|\bar{z}\phi''(z) + \psi'(z)|$ are invariants. As already remarked, the strain energy density for plane stress and plane strain cases given by equations (55) and (56) can be expressed in terms of these and they are respectively

$$W_0 = \frac{4(1-\nu)}{E}\left|\mathrm{Re}\,\phi'(z)\right|^2 + \frac{1+\nu}{E}\left|\bar{z}\phi''(z) + \psi'(z)\right|\left|z\overline{\phi''(z)} + \overline{\psi'(z)}\right|, \tag{93}$$

and

$$W_0 = \frac{(1+\nu)}{E}\left\{4(1-2\nu)\left|\mathrm{Re}\,\phi'(z)\right|^2 + \left|\bar{z}\phi''(z) + \psi'(z)\right|\left|z\overline{\phi''(z)} + \overline{\psi'(z)}\right|\right. . \tag{94}$$

CHAPTER VII

Elliptic Inclusion: Principal Strain

The complex variable technique of solving boundary value problems in elasticity theory has been outlined in chapter VI. As a preliminary example, we have solved the case of the circular inclusion in appendix II. As the next important case we take the case of an elliptic inclusion which may be of a material different from that of the outside material.

Consider an elliptic region with semi-axes a and b which would undergo a non-elastic deformation to an elliptic shape with semi-axes $a(1+\delta_a)$ and $b(1+\delta_b)$ in the absence of the matrix, the axes of the two ellipses remaining coincident. It is assumed that δ_a and δ_b fall within the proportional limit of the inclusion and no relative slipping takes place. Let us assume that the possible equilibrium boundary is an ellipse of semi-axes $a(1+\epsilon_1)$ and $b(1+\epsilon_2)$. Therefore, the elastic displacement of the boundary of the inclusion is from the ellipse of semi-axes $a(1+\delta_a)$ and $b(1+\delta_b)$ to that of $a(1+\epsilon_1)$ and $b(1+\epsilon_2)$.

The displacement field in the inclusion corresponding to the elastic boundary displacement is given by

$$U_x = (\epsilon_1 - \delta_a)\, x \,, \qquad U_y = (\epsilon_2 - \delta_b)\, y$$

whence the strains are, by equations (22)

$$\varepsilon_{xx} = (\epsilon_1 - \delta_a) \;, \quad \varepsilon_{yy} = (\epsilon_2 - \delta_b) \;, \quad \varepsilon_{xy} = 0 \tag{95}$$

and the dilation field is

$$\Delta = (\epsilon_1 + \epsilon_2 - \delta_a - \delta_b)$$

By making use of the stress-strain relations (24) and equations (95) the stress field in the inclusion is given by

$$P_{xx} = \lambda_1(\epsilon_1 + \epsilon_2 - \delta_a - \delta_b) + 2\mu_1(\epsilon_1 - \delta_a) \;,$$
$$P_{yy} = \lambda_1(\epsilon_1 + \epsilon_2 - \delta_a - \delta_b) + 2\mu_1(\epsilon_2 - \delta_b) \;, \qquad P_{xy} = 0 \;. \tag{96}$$

The values of strains and stresses given by equations (95) and (96) satisfy the equations of equilibrium, the compatibility relations and the boundary conditions. By uniqueness theorem in elasticity theory for a simply connected body, this is the solution.

By substituting the values of P_{xx} , P_{yy} and P_{xy} in equation (56), the strain energy density in the inclusion is obtained. It is given by, after some elementary simplification,

$$\frac{1}{2}\left[\lambda_1(\epsilon_1 + \epsilon_2 - \delta_a - \delta_b)^2 + 2\mu_1\left\{(\epsilon_1 - \delta_a)^2 + (\epsilon_2 - \delta_b)^2\right\}\right] . \tag{97}$$

Integrating equation (97) over the area of the inclusion we get the strain energy per unit thickness to be

$$W_I = \frac{\pi ab}{2}\left[\lambda_1(\epsilon_1 + \epsilon_2 - \delta_a - \delta_b)^2 + 2\mu_1\left\{(\epsilon_1 - \delta_a)^2 + (\epsilon_2 - \delta_b)^2\right\}\right] . \tag{98}$$

The displacement components of the interior boundary of the matrix are given by

$$u_x = \epsilon_1 x = \epsilon_1 \frac{z + \bar{z}}{2} \;, \quad u_y = \epsilon_2 y = \epsilon_2 \frac{z - \bar{z}}{2i} \;. \tag{99}$$

It may be remarked that unlike the inclusion, the strain field in the matrix cannot be deduced from the above set of equations. Since the boundary displacements are known, the problem can be

taken to be the second fundamental problem, already stated in chapter VI.

As already mentioned, this problem consists in finding the elastic field in a medium when the displacements are given at the surface. In plane elasticity this can be done by finding two complex potentials $\phi(z)$ and $\psi(z)$ throughout the region. In case of perforations other than unit circle, a mapping function is required which maps the region exterior to the boundary to the region within (or without) the circle. In the following analysis those mapping functions have been taken which map the outer region in z -plane to within the circle of unit radius.

The region outside the ellipse in the z -plane is mapped into a circle $|\zeta| \leq 1$ in the ζ -plane by the function

$$z \equiv \omega(\zeta) = R\left(m\zeta + \frac{1}{\zeta}\right) , \tag{100}$$

where $R=(a+b)/2$ and $m=(a-b)/(a+b)$. To distinguish the boundary values of ζ on the unit circle from those within it, ζ has been replaced by σ on the boundary. Obviously $\sigma\bar{\sigma} = 1$

It should be noted from equation (100) that as the point $\sigma = e^{i\theta}$ describes the circle $|\sigma| = 1$ in anti-clockwise direction, the point z traces the elliptic boundary in the clockwise direction. Thus the parametric equations of the ellipse will be

$$x = a\cos(-\theta) \equiv a\cos\theta, \quad y = b\sin(-\theta) \equiv -b\sin\theta$$

It may be noted that if $m=0$ in (100), $z=\omega(\zeta)$ maps the region exterior to a circle of radius R in z-plane to the interior of a unit circle in ζ-plane. On the otherhand if

$m \to 1$, the slit in the z-plane between $(2R, 0)$ and $(-2R, 0)$ is transformed into $|\zeta| \leq 1$. This is an important case from engineering point of view and refers for the case of a straight crack in an infinite material.

Substituting the value of ζ by equation (100) in equation (99) and making use of equation (87) the equation to determine the functions F and G is given by

$$k F(\sigma) - \left(m\sigma + \frac{1}{\sigma}\right) \overline{F'(\sigma)\left(\frac{\sigma^2}{m\sigma^2 - 1}\right)} - \overline{G(\sigma)} = A\sigma + \frac{B}{\sigma}, \tag{101}$$

where

$$A = \mu R\{(\epsilon_1 - \epsilon_2) + m(\epsilon_1 + \epsilon_2)\}, \quad B = \mu R\{(\epsilon_1 - \epsilon_2)m + (\epsilon_1 + \epsilon_2)\} \tag{102}$$

Multiplying both sides of equation (101) by $d\sigma/(\sigma - \zeta)2\pi i$ and then integrating round the contour Γ of the unit circle in ζ-plane, we get

$$\frac{k}{2\pi i}\int_\Gamma \frac{F(\sigma)\,d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int_\Gamma \frac{m\sigma^2 + 1}{\sigma(m - \sigma^2)} \frac{\overline{F'(\sigma)}}{(\sigma - \zeta)}\,d\sigma - \frac{1}{2\pi i}\int_\Gamma \frac{\overline{G(\sigma)}}{\sigma - \zeta}\,d\sigma$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{A\sigma\,d\sigma}{\sigma - \zeta} + \frac{1}{2\pi i}\int_\Gamma \frac{B\,d\sigma}{\sigma(\sigma - \zeta)}. \tag{103}$$

The proof that the second integral is zero and third can be taken to be equal to zero is given in books on elasticity theory [14]. Hence by equation (103) we get

$$k F(\zeta) = A\zeta, \quad \text{or} \quad F(\zeta) = \frac{A\zeta}{k}. \tag{104}$$

Taking the conjugate of equation (101) and integrating after multiplying by $d\sigma/(\sigma - \zeta)2\pi i$ it is easily seen that

$$G(\zeta) = \frac{\zeta(\zeta^2+m)}{1-m\zeta^2}\frac{A}{K} - B\zeta\,. \tag{105}$$

The calculation of the stresses everywhere involves the determination $\phi'(z)$, $\phi''(z)$ and $\psi'(z)$. These can be determined as follows:

$$\phi'(z) = \frac{d\phi}{dz} = \frac{dF(\zeta)}{d\zeta}\cdot\frac{d\zeta}{dz} = \frac{A}{k}\;\frac{\zeta^2}{R(m\zeta^2-1)}\;,$$

and similarly

$$\phi''(z) = \frac{-2A}{kR^2}\;\frac{\zeta^3}{(m\zeta^2-1)^3}\;,$$

$$\psi'(z) = \frac{d\psi}{dz} = \left\{\frac{A}{k}\;\frac{m+(3+m^2)\zeta^2-m\zeta^4}{(1-m\zeta^2)^2} - B\right\}\frac{\zeta^2}{R(m\zeta^2-1)} \tag{106}$$

Putting these values in equations (79) we get

$$p_{xx}+p_{yy} = 4\,\mathrm{Re}\,\phi'(z) = 4\,\mathrm{Re}\,\frac{A}{kR}\cdot\frac{\zeta^2}{(m\zeta^2-1)}\;,$$

$$p_{yy}-p_{xx}+2ip_{xy} = 2\left\{\bar{z}\,\phi''(z)+\psi'(z)\right\}$$

$$= \frac{2\zeta^2}{R(m\zeta^2-1)}\left[-\frac{2A}{k}\left(m\bar{\zeta}+\frac{1}{\bar{\zeta}}\right)\frac{\zeta}{(m\zeta^2-1)^2}+\left\{\frac{A}{k}\,\frac{m+(3+m^2)\zeta^2-m\zeta^4}{(1-m\zeta^2)^2}-B\right\}\right] \tag{107}$$

The stress field in the matrix is given by equations (107). Values of p_{xx} , p_{yy} and p_{xy} at the inner boundary distinguished by the superscript c are given by

$$p^c_{xx}+p^c_{yy} = 4\,\mathrm{Re}\,\frac{A}{kR}\cdot\frac{e^{2i\theta}}{(me^{2i\theta}-1)} = \frac{4A}{kR}\cdot\frac{m-\cos 2\theta}{m^2-2m\cos 2\theta+1}\;,$$

$$p^c_{yy} - p^c_{xx} + 2i p^c_{xy} = \frac{2e^{2i\theta}}{R(me^{2i\theta}-1)} \left[\frac{-2A(me^{-i\theta}+e^{i\theta})e^{i\theta}}{k(me^{2i\theta}-1)^2} + \left\{ \frac{A(m+(3+m^2)e^{2i\theta} - me^{4i\theta})}{k(1-me^{2i\theta})^2} - B \right\} \right]$$

$$= \frac{-2A}{kR} \frac{\{2\cos^2 2\theta - (3m+m^3)\cos 2\theta + (3m^2-1)\}}{(m^2-2m\cos 2\theta+1)^2} - \frac{2B}{R} \frac{m-\cos 2\theta}{(m^2-2m\cos 2\theta+1)}$$

$$+ \frac{2i}{R} \left[\frac{A(3m-m^3-2\cos 2\theta)\sin 2\theta}{k(m^2-2m\cos 2\theta+1)^2} - \frac{B\sin 2\theta}{(m^2-2m\cos 2\theta+1)} \right].$$

(108)

Separating the real and imaginary part of equations (108) and solving for p^c_{xx} , p^c_{yy} and p^c_{xy} we get

$$p^c_{xx} = \frac{A}{kR} \left[\frac{(4m+2)\cos^2 2\theta - (m^3+6m^2+3m+2)\cos 2\theta + (2m^3+3m^2+2m-1)}{(m^2-2m\cos 2\theta+1)^2} \right]$$

$$+ \frac{B}{R} \frac{m-\cos 2\theta}{(m^2-2m\cos 2\theta+1)} ,$$

$$p^c_{yy} = \frac{A}{kR} \left[\frac{(4m-2)\cos^2 2\theta + (m^3-6m^2+3m-2)\cos 2\theta + (2m^3-3m^2+2m+1)}{(m^2-2m\cos 2\theta+1)^2} \right]$$

$$- \frac{B}{R} \frac{m-\cos 2\theta}{(m^2-2m\cos 2\theta+1)} ,$$

$$p^c_{xy} = \frac{A}{kR} \frac{\{(3m-m^3)-2\cos 2\theta\}\sin 2\theta}{(m^2-2m\cos 2\theta+1)^2} + \frac{B}{R} \frac{\sin 2\theta}{(m^2-2m\cos 2\theta+1)}$$

(109)

The boundary values of the stress components have been found for two reasons. First, because they give the values of those external force intensities which account for a given displacement field. But the second and the more important reason is, that the strain energy density cannot be easily evalulated from the

expressions (107). We are, therefore, forced upon to apply Clapeyron's theorem, which relates the work done on the boundary to the strain energy in the body by

$$W_m = \frac{1}{2}\oint (P_{nx}u_x + P_{ny}u_y)\, ds,$$

$$= \frac{1}{2}\oint\Big[\overset{c}{p}_{xx}\cos(x,n)\,u_x + \overset{c}{p}_{yy}\cos(y,n)\,u_y + \overset{c}{p}_{xy}\big\{u_x\cos(y,n) + u_y\cos(x,n)\big\}\Big]\, ds,$$

$$= \frac{1}{2}\oint\Big[\overset{c}{p}_{xx}\epsilon_1 x\, dy - \overset{c}{p}_{yy}\epsilon_2 y\, dx + \overset{c}{p}_{xy}(\epsilon_2 y\, dy - \epsilon_1 x\, dx)\Big],$$

$$= \frac{R^2}{4}\oint\Big[\big\{(1+m^2)\epsilon_1 + (1-m^2)\epsilon_2\big\}\overset{c}{p}_{xy}\sin 2\theta - 2(1-m^2)\big\{\epsilon_1\overset{c}{p}_{xx}\cos^2\theta + \epsilon_2\overset{c}{p}_{yy}\sin^2\theta\big\}\Big]d\theta. \tag{110}$$

In the second line of these simplifications, we have put the values of P_{nx}, P_{ny} from equations (71). In the third, the values of u_x, u_y have been substituted from equations (99). It may be noted that use has been made of the relations $\cos(x,n) = \frac{dy}{ds}$, $\cos(y,n) = -\frac{dx}{ds}$. Finally, we have set $x = R\cos\theta$, $y = -R\sin\theta$.

Substitution of the values of $\overset{c}{p}_{xx}$, $\overset{c}{p}_{yy}$ and $\overset{c}{p}_{xy}$ from equations (109) in (110), will reveal that it involves some integrals, whose values are given in appendix III. Making use of these integrals in equation (110) we obtain

$$W_m = \frac{\pi ab}{2k}\left\{\frac{a}{b}\epsilon_1^2(k+1) + 2\epsilon_1\epsilon_2(k-1) + \frac{b}{a}\epsilon_2^2(k+1)\right\}\mu. \tag{111}$$

The total strain energy W of the system is the scalar sum of W_I given by equation (98) and W_m given by (111).

$$W = W_I + W_m = \frac{\pi ab}{2}\Big[\lambda_1(\epsilon_1+\epsilon_2-\delta_a-\delta_b)^2 + 2\mu_1\big\{(\epsilon_1-\delta_a)^2 + (\epsilon_2-\delta_b)^2\big\} + \frac{\mu}{k}\left\{\frac{a}{b}\epsilon_1^2(k+1) + 2\epsilon_1\epsilon_2(k-1) + \frac{b}{a}\epsilon_2^2(k+1)\right\}\Big] \tag{112}$$

Minimising W with respect to ϵ_1 and ϵ_2 (i.e., putting $\partial W/\partial\epsilon_1 = 0$ and $\partial W/\partial\epsilon_2 = 0$) we obtain the following set of simultaneous equations.

$$\epsilon_1\left\{\frac{\mu(k+1)a}{kb} + (\lambda_1+2\mu_1)\right\} + \left\{\frac{\mu(k-1)}{k} + \lambda_1\right\}\epsilon_2 - (\lambda_1+2\mu_1)\delta_a - \lambda_1\delta_b = 0,$$

$$\epsilon_1\left\{\frac{\mu(k-1)}{k} + \lambda_1\right\} + \epsilon_2\left\{\frac{\mu(k+1)b}{ka} + (\lambda_1+2\mu_1)\right\} - (\lambda_1+2\mu_1)\delta_b - \lambda_1\delta_a = 0.$$

Solving for ϵ_1 and ϵ_2 we obtain

$$\epsilon_1 = \frac{b\left[\delta_a\{b\mu(k+1)(\lambda_1+2\mu_1) + 4\mu_1 k(\lambda_1+\mu_1)a - \lambda_1\mu(k-1)a\} + \mu\delta_b\{\lambda_1(k+1)b - (\lambda_1+2\mu_1)(k-1)a\}\right]}{4ab\{\mu^2+\mu_1 k(\lambda_1+\mu_1)\} + \lambda_1\mu\{k(a-b)^2+(a+b)^2\} + 2\mu\mu_1(k+1)(a^2+b^2)},$$

$$\epsilon_2 = \frac{a\left[\delta_b\{a\mu(k+1)(\lambda_1+2\mu_1) + 4\mu_1 k(\lambda+\mu_1)b - \lambda_1\mu(k-1)b\} + \mu\delta_a\{\lambda_1(k+1)a - (\lambda_1+2\mu_1)(k-1)b\}\right]}{4ab\{\mu^2+\mu_1 k(\lambda_1+\mu_1)\} + \lambda_1\mu\{k(a-b)^2+(a+b)^2\} + 2\mu\mu_1(k+1)(a^2+b^2)}$$

(113)

It can be easily seen that $(\delta^2 W/\partial\epsilon_1^2)(\partial^2 W/\partial\epsilon_2^2) > (\partial^2 W/\partial\epsilon_1\,\partial\epsilon_2)^2$ which is true on physical grounds.

CHAPTER VIII

Principal Strains (Contd.)

The equilibrium interface has been obtained in terms of two parameters ϵ_1 and ϵ_2. It is based on the assumption that the inclusion of semi-axes (a, b) which tends to occupy the free surface of semi-axes $a(1+\delta_a)$, $b(1+\delta_b)$ will in equilibrium position also be an ellipse of semi-axes $a(1+\epsilon_1)$, $b(1+\epsilon_2)$ and further that the axes have not rotated. At first sight it appears to be a reasonable assumption which we shall prove to be correct. It may be emphasised that in all our present work, the continuity of the material is maintained throughout.

One may surmise that an ellipse of free surface size $2a(1+\delta) \times 2b(1+\delta)$ would be an ellipse of equilibrium size $2a(1+\epsilon) \times 2b(1+\epsilon)$. But this is not so as can be seen from the equations (113) by putting $\delta_a = \delta_b$. That is, in the general case a similar and similarly situated free surface would not have a similar and similarly situated equilibrium interface. This is an important result. The results obtained in the previous chapter, therefore, have to be substantiated by an independent check.

This consists in finding the normal and shearing stresses both for the inclusion and the matrix and proving their continuity. If this be proved, then since the values of ϵ_1 and ϵ_2 have been obtained from the equations of equilibrium, compatibility equations, stress-strain relations and boundary conditions and also satisfy the conditions of continuity of displacements, this would give the correct solution.

To prove the continuity of the normal and shear stresses it is necessary to determine the values of the constants A and B which have been introduced in equations (102). Substituting the values of ϵ_1 and ϵ_2 in terms of δ_a and δ_b in (102), these constants are given by

$$A = \frac{-2\mu a b k\left[\mu(\lambda_1+2\mu_1)(a\delta_b - b\delta_a) + \{\lambda_1\mu - 2\mu_1(\lambda_1+\mu_1)\}(a\delta_a - b\delta_b)\right]}{4ab\{\mu^2+\mu_1 k(\lambda_1+\mu_1)\} + \lambda_1\mu\{k(a-b)^2+(a+b)^2\} + 2\mu\mu_1(k+1)(a^2+b^2)},$$

$$B = \frac{2\mu a b\left[\mu(\lambda_1+2\mu_1)(b\delta_a + a\delta_b) + \{\lambda_1\mu + 2\mu_1(\lambda_1+\mu_1)k\}(a\delta_a + b\delta_b)\right]}{4ab\{\mu^2+\mu_1 k(\lambda_1+\mu_1)\} + \lambda_1\mu\{k(a-b)^2+(a+b)^2\} + 2\mu\mu_1(k+1)(a^2+b^2)}. \tag{114}$$

Denoting the normal, shear and hoop stresses by p_{nn}, p_{ns} and p_{ss} respectively and using the invariance relationship of the hydrostatic stress, we observe that

$$p_{nn} + p_{ss} = p_{xx} + p_{yy}. \tag{115}$$

When the (n, s) system of reference at a point is inclined at an angle β with (x, y) system of reference, the stresses are related by the relation

$$(p_{ss} - p_{nn} + 2ip_{ns}) = (p_{yy} - p_{xx} + 2ip_{xy})\, e^{-2i\beta}. \tag{116}$$

From the above-mentioned two relations (115) and (116) we evaluate

$$p_{nn} = \frac{p_{xx}+p_{yy}}{2} + \frac{p_{xx}-p_{yy}}{2}\ \operatorname{Cos} 2\beta + p_{xy}\operatorname{Sin} 2\beta\ ,$$

$$p_{ns} = -\frac{p_{xx}-p_{yy}}{2}\ \operatorname{Sin} 2\beta + p_{xy}\operatorname{Cos} 2\beta\ ,$$

$$p_{ss} = \frac{p_{xx}+p_{yy}}{2} - \frac{p_{xx}-p_{yy}}{2}\ \operatorname{Cos} 2\beta - p_{xy}\operatorname{Sin} 2\beta\ .$$

(117)

We first evaluate these normal shear and hoop stresses for the inclusion. Putting the values of P_{xx} , P_{yy} , and P_{xy} from equations (96) and substituting the values of ϵ_1 and ϵ_2 from equations (113) and making use of the expressions for A and B , it is readily seen that

$$P^{b}_{nn} = \frac{-\{A(a-b)+B(a+b)k\}-\{A(a+b)+B(a-b)k\}\ \operatorname{Cos}2\beta}{2kab}\ ,$$

$$P^{b}_{ns} = \frac{\{A(a+b)+B(a-b)k\}\ \operatorname{Sin}2\beta}{2kab}\ ,$$

$$P^{b}_{ss} = \frac{-\{A(a-b)+B(a+b)k\}+\{A(a+b)+B(a-b)k\}\ \operatorname{Cos}2\beta}{2kab}\ ,$$

(118)

where the superscript b refers to the boundary values of the stresses at equilibrium interface.

Similarly the values of p^{c}_{xx} , p^{c}_{yy} and p^{c}_{xy} for the matrix can be determined by equations (109) and (114). Making use of equations (117) and the trigonometric relations for the

ellipse

$$\cos 2\beta = \frac{-\{(a^2-b^2)-(a^2+b^2)\cos 2\theta\}}{(a^2+b^2)-(a^2-b^2)\cos 2\theta} \,,$$

$$\sin 2\beta = \frac{-2ab \sin 2\theta}{(a^2+b^2)-(a^2-b^2)\cos 2\theta} \,, \tag{119}$$

we obtain

$$p^{b}_{nn} = \left[-\{A(a-b)+B(a+b)k\} - \{A(a+b)+Bk(a-b)\}\cos 2\beta \right] \frac{1}{2kab} \,,$$

$$p^{b}_{ns} = \frac{1}{2kab}\{A(a+b)+Bk(a-b)\} \sin 2\beta \,,$$

$$p^{b}_{ss} = \frac{-1}{2kab}\left[\{3A(a-b)-Bk(a+b)\} - Bk(a-b) \cos 2\beta + 3A(a+b)\cos 2\beta\right]. \tag{120}$$

It can be seen by comparing the equations (118) and (120) that the normal stresses P^{b}_{nn} , p^{b}_{nn} and the shear stresses P^{b}_{ns} , p^{b}_{ns} are continuous. It may be remarked here that these results for the particular case, when the elastic properties of the inclusion and matrix are the same and $\delta_a = \delta_b = \delta$ agree with the values of the boundary stresses determined by Eshelby[4].

The hoop stresses are discontinuous at the interface. It may be seen from the expressions for P^{b}_{ss} , p^{b}_{ss} that when $\delta_a \geqslant 0$ and $\delta_b \geqslant 0$, p^{b}_{ss} is positive and tensile and P^{b}_{ss} is negative and compressive. This is obvious on physical grounds also. ($\delta_a = \delta_b = 0$ is a trivial case as there will be no stress

field.) The jump in the hoop stress is given by

$$p_{ss}^{t} - P_{ss}^{t} = -\left[\left\{A(a-b) - B(a+b)k\right\} + 2A(a+b)\cos 2\beta\right]\frac{1}{kab} .$$

It is interesting to draw lines of maximum shear stress in the matrix. These have been drawn in fig. 2 for the special case when $\delta_a = \delta_b = \delta$ for axial ratios $a/b = 2$ and 5 when the inclusion and matrix are of the same materials.

The potential functions F and G have been evaluated for the matrix. Making use of these from equations (104) and (105) together with equation (100) in equations(85), the values of u_x and u_y can be evaluated at any point in the matrix. However at the interface the equilibrium boundary has been taken to be an ellipse of semi-axes $a(1+\epsilon_1)$, $b(1+\epsilon_2)$. Substitution of the values of ϵ_1 , ϵ_2 gives the shape of the interface. In table 1, we give the values of ϵ_1 , ϵ_2, in terms of δ_a, δ_b when the Poisson's ratio is $1/4$, or $1/3$ or $1/2$ both for the inclusion and matrix materials, and their Young's modulii are in the ratios 0 , $1/3$, 1 , 3 and ∞ . This accounts for different tensile strengths of the inclusion. This table is given for some axial ratios of the elliptic inclusion.

If the inclusion is of a harder material, then its non-elastic deformation would have a dominant influence on the equilibrium interface and the stress distribution. It may be noted that the equilibrium position depends upon the axial ratios and the relative stress-free displacements and the properties of the materials. In fig. 3 the schematic repre-

sentation for the equilibrium interface is given for some ratios of δ 's for $a/b=5$. It may be remarked that in the analysis given in the previous chapter, nothing precludes the possibility of negative values of δ 's. Hence if we take $\delta_a=\delta$ and $\delta_b=-\delta$, the case of a shear strain can be obtained.

The strain energy in the inclusion and matrix can be determined by just setting the values of ϵ_1 and ϵ_2 from equations (113) in equation (98) and in equation (111) respectively. It can be easily seen from equations (113) that if the inclusion is circular, i.e., $a=b$, we get

$$\epsilon_1 = \frac{\delta_a\{(\lambda_1+\mu_1)(\mu+2\mu_1 k)+\mu\mu_1 k\}+\delta_b\mu\{\lambda_1-\mu_1(k-1)\}}{2(\mu+\mu_1 k)(\lambda_1+\mu_1+\mu)},$$

$$\epsilon_2 = \frac{\delta_b\{(\lambda_1+\mu_1)(\mu+2\mu_1 k)+\mu\mu_1 k\}+\delta_a\mu\{\lambda_1-\mu_1(k-1)\}}{2(\mu+\mu_1 k)(\lambda_1+\mu_1+\mu)}.$$

By setting $\delta_a=\delta_b=\delta$ we recover the result given in chapter V,

$$\epsilon_1=\epsilon_2=\epsilon=\frac{(\lambda_1+\mu_1)\delta}{(\lambda_1+\mu_1+\mu)}.$$

If the inclusion is a slender one i.e., $b/a \to 0$, the corresponding results of ϵ_1 and ϵ_2 can be evaluated by equations (113). These are

$$\epsilon_1=0,\quad \epsilon_2=\delta_b+\frac{\lambda_1\delta_a}{(\lambda_1+2\mu_1)}.$$

Further if the inclusion is incompressible, i.e., $\lambda_1 \to \infty$ (μ_1 remaining finite)

$$\epsilon_1 = \frac{b[\delta_a\{b\mu(k+1)+4a\mu_1 k-\mu(k-1)a\}+\mu\delta_b\{(a+b)-(a-b)k\}]}{4ab\mu_1 k+\mu\{k(a-b)^2+(a+b)^2\}} ,$$

$$\epsilon_2 = \frac{a[\delta_b\{a\mu(k+1)+4b\mu_1 k-\mu(k-1)b\}+\mu\delta_a\{(a+b)+(a-b)k\}]}{4ab\mu_1 k+\mu\{k(a-b)^2+(a+b)^2\}} .$$

On the otherhand if both λ_1 and μ_1 tend to infinity, i.e., the inclusion is rigid, we get $\epsilon_1 = \delta_a$, $\epsilon_2 = \delta_b$. If the matrix is incompressible i.e., $k = 3-4\nu = 1$ (since $\nu = 1/2$)

$$\epsilon_1 = \frac{b[\delta_a\{b\mu(\lambda_1+2\mu_1)+2\mu_1(\lambda_1+\mu_1)a\}+2\lambda_1\mu b\delta_b]}{2ab\{\mu^2+\mu_1(\lambda_1+\mu_1)\}+\mu(\lambda_1+2\mu_1)(a^2+b^2)} ,$$

$$\epsilon_2 = \frac{a[\delta_b\{a\mu(\lambda_1+2\mu_1)+2\mu_1(\lambda_1+\mu_1)b\}+2\lambda_1\mu a\delta_a]}{2ab\{\mu^2+\mu_1(\lambda_1+\mu_1)\}+\mu(\lambda_1+2\mu_1)(a^2+b^2)}$$

If the matrix is rigid, $\epsilon_1 = 0$, $\epsilon_2 = 0$ which is correct on physical grounds as well.

CHAPTER IX

Elliptic Inclusion: Shear Strain

In this chapter we consider the case of an elliptic inclusion which undergoes spontaneously a non-elastic deformation in which the displacement components are $\gamma_a y$ parallel to x-axis and $\gamma_b x$ parallel to y-axis. (γ_a and γ_b are within proportional limits.) This means that the free surface will be an ellipse whose parametric equations will be

$$x = a \cos\theta + \gamma_a b \sin\theta, \quad y = b \sin\theta + \gamma_b a \cos\theta.$$

The major axis of this will be inclined to x-axis by a small angle α given by

$$\alpha = \frac{a^2\gamma_b + b^2\gamma_a}{b^2 - a^2} \, .$$

The lengths of the major and minor axes of this ellipse will respectively be

$$2\left[\frac{a^2}{(a^2-b^2)}\left\{a^2(\gamma_a+\gamma_b)^2 + (a^2-b^2)(1-\gamma_a^2)\right\}\right]^{\frac{1}{2}},$$

$$2\left[\frac{b^2}{(a^2-b^2)}\left\{(a^2-b^2)(1-\gamma_b^2) - b^2(\gamma_a+\gamma_b)^2\right\}\right]^{\frac{1}{2}},$$

neglecting fourth and higher powers of γ's. If the second order effects are also neglected, as we do in linear elasticity theory, then the major and minor axes will be $2a$ and $2b$ respectively.

The possible equilibrium boundary is assumed to be an ellipse which again is measured from the initial size of the

hole in the matrix and is such that it can be obtained by giving a displacement,

$$u_x = \gamma_1 y \ , \qquad u_y = \gamma_2 x \ , \tag{121}$$

to its boundary. We shall show that this assumption regarding the equilibrium interface is correct. We shall determine the values of γ_1 and γ_2 in terms of γ_a, γ_b, a and b and the elastic properties of the materials. It may incidentally be remarked that γ_1, γ_2 would depend upon the axial ratios and $(\gamma_a + \gamma_b)$. This latter observation follows from the fact that the non-elastic displacements $\gamma_a y$ and $\gamma_b x$ can be written as

$$\gamma_a y = \left\{ (\gamma_a + \gamma_b) + (\gamma_a - \gamma_b) \right\} \ y/2 \ ,$$

$$\gamma_b x = \left\{ (\gamma_a + \gamma_b) - (\gamma_a - \gamma_b) \right\} \ x/2 \ ,$$

and the second terms on the right hand side of $\gamma_a y$ and $\gamma_b x$ account for a rigid body rotation.

The elastic displacement of the boundary of the inclusion has components

$$(\gamma_1 - \gamma_a)\, y \ , \qquad (\gamma_2 - \gamma_b)\, x \ .$$

By the same type of reasoning which we used for principal strains, it can be seen that the displacement field at any point in the inclusion is given by

$$u_x = (\gamma_1 - \gamma_a)\, y \ , \qquad u_y = (\gamma_2 - \gamma_b)\, x \ . \tag{122}$$

By making use of the strain-displacement relations (22)

$$\varepsilon_{xx} = 0 \ , \ \varepsilon_{yy} = 0 \ , \ \varepsilon_{xy} = \tfrac{1}{2}\,(\gamma_1 + \gamma_2 - \gamma_a - \gamma_b) \ . \tag{123}$$

These are related to the stresses by Hooke's law and therefore, the stress field in the inclusion

$$P_{xx} = 0 \;,\quad P_{yy} = 0 \;,\quad P_{xy} = \mu_1(\gamma_1 + \gamma_2 - \gamma_a - \gamma_b) \tag{124}$$

Equations (122) satisfy the boundary conditions. Also the equations (123) and (124) satisfy the equations of equilibrium and the compatibility equations. Therefore, these are the true stresses and strains corresponding to the elastic displacements of the inclusion.

By putting these values of stresses in equation (56) the strain energy density is determined to be

$$\frac{1}{2}\;\mu_1(\gamma_1 + \gamma_2 - \gamma_a - \gamma_b)^2$$

which when integrated over the area of the inclusion yields its strain energy per unit thickness to be

$$W_I = \frac{\pi a b \mu_1}{2}\,(\gamma_1 + \gamma_2 - \gamma_a - \gamma_b)^2 \tag{125}$$

The boundary displacement of the matrix is given by equation (121) and is

$$u_x = \gamma_1 y = \gamma_1 \frac{z - \bar{z}}{2i} \;,\qquad u_y = \gamma_2 x = \gamma_2 \frac{z + \bar{z}}{2}$$

It is not possible to assume the displacement field in the matrix from this set of boundary displacements. Hence to determine the stress-field etc., in the matrix for the boundary conditions (121), the complex variable method utilised for the principal strain problem (chapter VII) is used.

The mapping function

$$z = \omega(\zeta) = R\left(m\zeta + \frac{1}{\zeta}\right) ,$$

given by equation (100) is again used to map the region outside the ellipse in the z-plane into a unit circle $|\zeta| = 1$ in the ζ-plane. We substitute for z in equations (121) in terms of ζ from the mapping function given above and put $\zeta = \sigma$ on the boundary. This boundary value of u_x, u_y in terms of σ is substituted in equation (87). It may be noted that in the right hand side of this equation, the boundary value of z has already been replaced by its boundary value in terms of σ. Thus the functions F and G are to be obtained from the equation

$$k\,F(\sigma) - \frac{m\sigma^2+1}{\sigma(m-\sigma^2)}\,\overline{F'(\sigma)} - \overline{G(\sigma)} = i\left(A_1\sigma + \frac{B_1}{\sigma}\right), \tag{126}$$

where

$$A_1 = \left\{(\gamma_2-\gamma_1)m + (\gamma_2+\gamma_1)\right\}\mu R,\quad B_1 = \mu R\left\{(\gamma_2-\gamma_1) + (\gamma_2+\gamma_1)m\right\}. \tag{127}$$

The functions $F(\zeta)$ and $G(\zeta)$ have been determined from equation (126) by forming the integro-differential equation and solving them by the method described in chapter VI and applied in chapter VII. These are found to be

$$F(\zeta) = \frac{iA_1}{k}\,\zeta,$$

$$G(\zeta) = \frac{iA_1\zeta(\zeta^2+m)}{k(1-m\zeta^2)} + iB_1\zeta. \tag{128}$$

The stress components p_{xx}, p_{yy} and p_{xy} can now be determined by first determining $\phi'(z)$, $\psi'(z)$ from the functions

$F(\zeta)$ and $G(\zeta)$ making use of an inverse transformation from ζ to z. Relations between these functions are already known by equations (106). These are given by

$$\phi'(z) = \frac{iA_1}{kR}\frac{\zeta^2}{(m\zeta^2-1)}, \quad \phi''(z) = \frac{-2iA_1}{kR^2}\frac{\zeta^3}{(m\zeta^2-1)^3},$$

$$\psi'(z) = \left\{\frac{iA_1}{k}\frac{m+(3+m^2)\zeta^2-m\zeta^4}{(1-m\zeta^2)^2} + iB_1\right\}\frac{\zeta^2}{R(m\zeta^2-1)}. \tag{129}$$

Substitution of these values in equations (79) yields for the stress field in the matrix

$$p_{xx}+p_{yy} = 4\,\mathrm{Re}\,\frac{iA_1}{kR}\frac{\zeta^2}{(m\zeta^2-1)},$$

$$p_{yy}-p_{xx}+2ip_{xy} = \frac{2i\zeta^2}{R(m\zeta^2-1)}\left[\frac{-2A_1}{k}\left(m\bar{\zeta}+\frac{1}{\bar{\zeta}}\right)\frac{\zeta}{(m\zeta^2-1)^2} + \left\{\frac{A_1}{k}\frac{m+(3+m^2)\zeta^2-m\zeta^4}{(1-m\zeta^2)^2}+B_1\right\}\right]. \tag{130}$$

The strain energy in the matrix is determined by the same methods which were used in the principal strain problem, i.e., by the theorem relating the work done at the boundary to the strain-energy. To calculate the work done at the boundary, the boundary tractions are first determined. For these the values of the boundary stresses are evaluated. These are determined by putting $\zeta = \sigma = e^{i\theta}$ in (130). By solving, we get

$$p^c_{xx} = \frac{A_1\{(2+3m+2m^2-m^3)-2(2m+1)\cos 2\theta\}\sin 2\theta}{kR(m^2-2m\cos 2\theta+1)^2}$$

$$-\frac{B_1 \sin 2\theta}{R(m^2-2m\cos 2\theta+1)},$$

$$p^c_{xy} = \frac{-A_1\{2\cos^2 2\theta-(3m+m^3)\cos 2\theta+(3m^2-1)\}}{kR(m^2-2m\cos 2\theta+1)^2}$$

$$+\frac{B_1(m-\cos 2\theta)}{R(m^2-2m\cos 2\theta+1)},$$

$$p^c_{yy} = \frac{A_1\{(2-3m+2m^2+m^3)+2(1-2m)\cos 2\theta\}\sin 2\theta}{kR(m^2-2m\cos 2\theta+1)^2}$$

$$+\frac{B_1 \sin 2\theta}{R(m^2-2m\cos 2\theta+1)}.$$

(131)

With the help of the equations (71) the values of the surface tractions P_{nx} and P_{ny} are given by

$$P_{nx} = p^c_{xx}\cos(x,n)+p^c_{xy}\cos(y,n), \quad P_{ny} = p^c_{xy}\cos(x,n)+p^c_{yy}\cos(y,n).$$

Making use of the surface displacements by equations (121), and noting that the strain energy is

$$W = \frac{1}{2}\oint (P_{nx}u_x + P_{ny}u_y)\,ds,$$

per unit height we obtain the strain energy W_m in the matrix.

After changing x, y in u_x, u_y, $\cos(x,n)$, $\cos(y,n)$, ds in terms of the single variable θ (as in case of principal strain) we obtain

$$W_m = \frac{1}{2}\int_0^{2\pi} R^2\Big[p^c_{xx}\gamma_1 \frac{(1-m)^2}{2} \sin 2\theta + p^c_{yy}\gamma_2 \frac{(1+m)^2}{2} \sin 2\theta - p^c_{xy}(1-m^2)\big\{\gamma_2\cos^2\theta + \gamma_1 \sin^2\theta\big\}\Big]\, d\theta . \tag{132}$$

Putting the values of p^c_{xx} , p^c_{yy} and p^c_{xy} from equations (131) in equation (132) and integrating, we shall obtain

$$W_m = \frac{\mu\pi ab}{2k}\left\{\frac{b}{a}(k+1)\gamma_1^2 + 2(1-k)\gamma_1\gamma_2 + \frac{a}{b}(k+1)\gamma_2^2\right\} . \tag{133}$$

At this stage, we shall have to make use of some integrals which are given in appendix III.

The total strain energy in the medium from equations (125) and (133) is

$$W = W_I + W_m = \frac{\pi ab}{2}\Big[\frac{\mu}{k}\Big\{\frac{b}{a}(1+k)\gamma_1^2 + 2(1-k)\gamma_1\gamma_2 + \frac{a}{b}(1+k)\gamma_2^2\Big\} + \mu_1(\gamma_1+\gamma_2-\gamma_a-\gamma_b)^2\Big] . \tag{134}$$

Minimising W with respect to γ_1 and γ_2, the following set of linear simultaneous equations are obtained.

$$\{\mu(1+k)b + ak\mu_1\}\gamma_1 + a\gamma_2\{(1-k)\mu + \mu_1 k\} - \mu_1 k a(\gamma_a+\gamma_b) = 0$$

$$\{\mu(1+k)a + bk\mu_1\}\gamma_2 + b\gamma_1\{(1-k)\mu + \mu_1 k\} - \mu_1 k b(\gamma_a+\gamma_b) = 0$$

The values of γ_1 and γ_2 are evaluated from these to be

$$\gamma_1 = \frac{a\mu_1\{(a+b)(k+1) - 2b\}(\gamma_a + \gamma_b)}{4ab(\mu - \mu_1) + \mu_1(a+b)^2(k+1)}$$

$$\gamma_2 = \frac{b\mu_1\{(a+b)(k+1) - 2a\}(\gamma_a + \gamma_b)}{4ab(\mu - \mu_1) + \mu_1(a+b)^2(k+1)}$$

(135)

Substitution of γ_1 and γ_2 in relevant equations determines the elastic field, in the inclusion and matrix materials.

CHAPTER X

Shear Strain (Contd.)

As has been mentioned in chapter VIII, it is necessary to substantiate the results by satisfying the usual boundary conditions. This consists in proving the continuity of the normal and shear stresses at the interface, and also of the continuity of the boundary displacements.

The cartesian components of stresses in the inclusion are obtained by setting the values of γ_1 , γ_2 from equations (135) in (124). As the normal and shear stresses are to be continuous at the interface, it is useful to express these stress components in terms of the normal, shear and hoop stress components. We again use the formulae (117) which relates the normal, shear and hoop stresses to cartesian stress components. These are found to be

$$P_{nn}^{b} = \frac{-4\mu\mu_1 ab(\gamma_a+\gamma_b)\sin 2\beta}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)},$$

$$P_{ns}^{b} = \frac{-4\mu\mu_1 ab(\gamma_a+\gamma_b)\cos 2\beta}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)},$$

$$P_{ss}^{b} = \frac{4\mu\mu_1 ab(\gamma_a+\gamma_b)\sin 2\beta}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)}.$$

(136)

Substitution of the values of γ_1 and γ_2 from equations (135) in equations (131) determines the values of the cartesian stress components at the interface in the matrix. In expressions

on the right hand side of equations (131) constants A_1 and B_1 are involved. Their values are obtained by substituting the values of γ_1, γ_2 from equations (136) into (127). They are

$$A_1 = \frac{2\mu\mu_1 ab(a+b)(\gamma_a+\gamma_b)k}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)},$$

$$B_1 = \frac{-2\mu\mu_1 ab(a+b)(\gamma_a+\gamma_b)}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)}.$$

Utilising these values to determine p^c_{xx}, p^c_{yy} and p^c_{xy} and making use of equations (117) and the trigonometrical relations of equations (119), the normal, shear and hoop stresses in the matrix at the interface are given by

$$p^{b}_{nn} = \frac{-4\mu\mu_1 ab(\gamma_a+\gamma_b)\sin 2\beta}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)},$$

$$p^{b}_{ns} = \frac{-4\mu\mu_1 ab(\gamma_a+\gamma_b)\cos 2\beta}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)},$$

$$p^{b}_{ss} = \frac{-4\mu\mu_1(a^2+ab+b^2)(\gamma_a+\gamma_b)\sin 2\beta}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)}.$$

(137)

It may be seen by equations (136) and (137) that the normal and shear stresses are continuous. It may be remarked that normal and hoop stresses will be zero at the ends of the major and minor axes both for the inclusion as well as for the matrix. The shear stresses at these points are the maximum and bear opposite sense at these points. This is an important result

which states that at the ends of the major or minor axes, the principal stresses are at an angle of $45°$ to the axes of the ellipse. By the knowledge of the principal stresses or the maximum shear stress at these points, yielding may be predicted.

The effect of shear stresses rapidly reduces as we travel along the interface from the ends of the major axis towards the ends of the minor axis. It actually becomes zero at $\beta = \pi/4$. Then it increases in the opposite sense at a slower rate, to reach the same magnitude at the tips of the minor axis. For an inclusion with small minor axis, the angle $\beta = \pi/4$ is attained fairly near to the tips of the major axis. Thus the shear stress has predominant influence on the deformation at the ends of the major axis. This explains, for example, the twinning effects actually seen in the formation of alloys, where the shearing stresses are maximum at the tips. Such phenomenon are extensively given in the literature on Physical Metallurgy, [15] " In either of these cases, as the slip planes rotate away from a $45°$ angle .... than on the active ones. "

It might be remarked that the hoop stress will be much greater in the matrix than that in the inclusion as $b/a \to 0$ except at the ends of axes where they are equal to zero. The ratio of the hoop stress in matrix to that in the inclusion is given by

$$-\frac{(a^2 + ab + b^2)}{ab}$$

The negative sign indicates that the hoop stresses will have opposite sense in the two materials. Further if $b|a \to 0$ this ratio is approximately equal to $-a|b$. This implies that except at the tips of the axes, the hoop stresses in the matrix and the inclusion are numerically in the ratio of the lengths of the major and minor axes. For $a|b = 1$, the hoop stress in the matrix is three times that of the inclusion.

Further it may be seen from the expressions for the hoop stress that, for matrix, these are either compressive or tensile in the first and third quadrants of the ellipse and tensile or compressive in the second and fourth quadrants depending upon whether $(\gamma_a + \gamma_b)$ is positive or negative. Reverse remarks apply to the case of inclusion. The jump in the hoop stress is given by

$$p_{ss}^{L} - p_{ss}^{L} = \frac{-4\mu\mu_1(a+b)^2(\gamma_a+\gamma_b)\sin 2\beta}{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)} .$$

The displacement field in the inclusion is obtained by putting the values of γ_1 and γ_2 in the equations (122). As regards the matrix, it is found by the relation

$$2\mu(u_x + iu_y) = k\,\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi(z)} ,$$

given in equation (85). The function $\phi'(z)$ is known by equations (129) and the functions $\phi(z)$ and $\psi(z)$ can be deduced by inverse transformation of $F(\zeta)$ and $G(\zeta)$ given in equations (128). We shall not evaluate the displacement fields, in the matrix or the inclusion, as it does not reveal any interesting feature.

The equilibrium interface is interesting and can be obtained by equations (121) after putting the values of γ_1 and γ_2 from equations (135). It can be readily checked that the net displacement at the interface for the inclusion is the same as that of the matrix. This establishes the continuity of the displacements. It may, however, be emphasised that the net displacement in the inclusion is to be measured from its initial size (i.e., the size before it undergoes the non-elastic deformation).

The values of $\gamma_1 / (\gamma_a + \gamma_b)$, $\gamma_2 / (\gamma_a + \gamma_b)$ corresponding to some values of axial ratio a/b, for materials having different Young's modulii are given in table 2. The Poisson's ratio has been taken to be the same to simplify the calculation. As already stated, even when the Poisson's ratio are different, the formulae (135) may be applied to find the equilibrium boundary.

The strain energy in the inclusion is found by putting the values of γ_1 and γ_2 in equation (125). This is given by

$$W_I = \frac{8a^3b^3\mu^2\mu_1(\gamma_a+\gamma_b)^2\pi}{\{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)\}^2}$$

Similarly, the strain energy in the matrix is obtained from equation (133) and is

$$W_m = \frac{2\pi(k+1)a^2b^2\mu_1^2\mu\{(a+b)^2k+(a-b)^2\}(\gamma_a+\gamma_b)^2}{\{4ab(\mu-\mu_1)+\mu_1(a+b)^2(k+1)\}^2}.$$

The total energy in the inclusion and matrix is thus

$$W = \frac{2\pi a^2 b^2 \mu \mu_1 (\gamma_a+\gamma_b)^2 [4ab\mu + (k+1)\mu_1\{(a+b)^2 k + (a-b)^2\}]}{\{4ab(\mu-\mu_1) + \mu_1 (a+b)^2 (k+1)\}^2} .$$

The ratio of the energy in the matrix to that in the inclusion is

$$\frac{W_m}{W_I} = \frac{\mu_1\{(a+b)^2 k + (a-b)^2\}(k+1)}{4ab\mu} .$$

When the rigidity modulii of the inclusion and matrix are equal and the Poisson's ratio of the matrix material is $1/2$, then W_m/W_I is given by

$$\frac{(a^2+b^2)}{ab}$$

which states that the energy in the matrix is greater than that in the inclusion in that case.

The results given in equations(135) lead to many special cases. When the inclusion is circular $a=b$ we obtain

$$\gamma_1 = \gamma_2 = \frac{\mu_1 k(\gamma_a+\gamma_b)}{2(\mu+\mu_1 k)} .$$

On the other hand if the inclusion is slender and $b/a \to 0$,

$$\gamma_1 = (\gamma_a+\gamma_b) , \quad \gamma_2 = 0 .$$

Further if the inclusion is rigid, $\mu_1 \to \infty$

$$\gamma_1 = \frac{a\{(a+b)(k+1)-2b\}(\gamma_a+\gamma_b)}{k(a+b)^2+(a-b)^2} ,$$

$$\gamma_2 = \frac{b\{(a+b)(k+1)-2a\}(\gamma_a+\gamma_b)}{k(a+b)^2+(a-b)^2} .$$

In this case the non-elastic strains in the inclusion given by $(\gamma_a+\gamma_b)$ remains unaltered. However the initial rigid body rotation $(\gamma_a-\gamma_b)$ is changed to $(\gamma_1-\gamma_2)$ which is different from $(\gamma_a-\gamma_b)$. If the matrix is in-compressible $k=1$ the values of γ_1 and γ_2 will become

$$\gamma_1 = \frac{a^2\mu_1(\gamma_a+\gamma_b)}{2ab\mu+\mu_1(a^2+b^2)} \quad , \quad \gamma_2 = \frac{b^2\mu_1(\gamma_a+\gamma_b)}{2ab\mu+\mu_1(a^2+b^2)} \,.$$

But if the matrix is rigid, the inclusion remaining elastic, then $\gamma_1 = 0$, $\gamma_2 = 0$

We might conclude that the case when the inclusion undergoes a non-elastic perturbation, which can be isolated as the sum of the principal and shear strains i.e., of the form $\delta_a x+\gamma_a y$ parallel to x-axis and $\delta_b y+\gamma_b x$ parallel to y-axis. Then the equilibrium boundary is obtained by giving a displacement to the boundary of the hole

$$u_x = \epsilon_1 x + \gamma_1 y \quad , \quad u_y = \epsilon_2 y + \gamma_2 x$$

where the values of ϵ_1 and ϵ_2 , γ_1 and γ_2 are given by the results in equations (113) and (135). The elastic field in the matrix and the inclusion can be obtained by superposing the relevant stress-strain field obtained in chapters VII and IX.

So far we have obtained solutions for the problem of inclusion in otherwise unstressed matrix. Inclusion was either of spherical shape or had a circular or elliptic cross-section. In the following five chapters we discuss the more interesting and technically important problem of the inclusion in matrix which is otherwise also stressed.

CHAPTER XI

Spherical Inclusion in Stressed Matrix

The effect of the external field applied to the matrix on the stress-strain field and the equilibrium boundary was discussed by Eshelby [16] . In this case he distinguished two types of problems namely those of (1) transformation of inclusion problem (2) inhomogeneity problem.

Transformation of inclusion problems are of the type which we have dealt with in previous chapters, but in Eshelby's analysis the elastic constants of the inclusion are the same as of outside material. Again according to him the inhomogeneity is a material having elastic properties different from those of the outside region in which it is embedded. Left to itself, it creates no stress field in the medium. But if external forces are applied to outside region, the presence of the inhomogeneity perturbs the elastic field. By ingenious arguments Eshelby accounts for the perturbations of an external stress field due to inhomogeneity, to the problem of transformation of inclusion. This method appears to be cumbersome and does not easily lead to explicit solutions.

Here we modify the methods enunciated and exemplified in the previous chapters and obtain explicit solutions for some problems. Further we do not distinguish between so called inhomogeneity and the transformation problems. An inclusion can be and in fact has been taken to be of a material which is different from that of the outside material and may undergo a transformation. Thus it includes both the problems of

' inhomogeneity ' and ' transformation of inclusion ' as described by Eshelby.

The method consists in finding the displacement at the inner boundary and the stress-strain field due to the external forces. We term this to be the free state of the matrix, (as in this condition we imagine that the matrix is free of inclusion). The free state of the matrix is indicated in fig. 4. It may be noted that the inner boundary of the hole in the free state would be different from what it was before the matrix was loaded at the external region. We now imagine, that the inclusion is present and tries to attain its free surface. Because of the mutual constraints, neither the inclusion is able to attain its free surface nor the matrix is able to retain its free state. There will be an equilibrium interface which will be different from the free surface as well as free state. The problem is to find the elastic field in the inclusion and the resultant field in the matrix.

The energy in the inclusion is calculated from the elastic displacements of its boundary from its free surface to the equilibrium position. The energy in the matrix is calculated from the elastic displacements which it undergoes from its free state to the equilibrium position. Thus the total energy of the superposing system is obtained and is minimised. This determines the equilibrium position, from which the elastic field in the inclusion can be evaluated. As regards the matrix the actual stress, strain and displacement field is obtained by the superposition of the field due to the perturbation of the inclu-

sion over that due to the external forces.

We exemplify the above analysis with reference to a solid spherical inclusion embedded in a concentric, finite spherical region. The initial radius of the inclusion is a. It tends to undergo a dimensional change to a concentric sphere of radius $a(1+\delta)$ in the absence of the surrounding matrix. Let the outer radius of the matrix be b. A uniform radial pressure P_0 is applied at the outer surface of the matrix.

On symmetry considerations, the equilibrium shape of the inclusion shall be a concentric sphere of radius $a(1+\epsilon)$. As far as the inclusion is concerned, its elastic boundary displacements are $a(\epsilon-\delta)$ and from elementary elasticity theory, it can be shown that the strain components are

$$\mathcal{E}_{rr}=\mathcal{E}_{\theta\theta}=\mathcal{E}_{\phi\phi}=(\epsilon-\delta)\,, \quad \mathcal{E}_{r\theta}=\mathcal{E}_{\theta\phi}=\mathcal{E}_{\phi r}=0\,, \tag{138}$$

and the dilation field is $\Delta=3(\epsilon-\delta)$. We use the same notations for stresses and strains as we used in previous chapters. By the stress-strain relations (10), we get the stress components to be

$$P_{rr}=P_{\theta\theta}=P_{\phi\phi}=3K_1(\epsilon-\delta)\,, \quad P_{r\theta}=P_{\theta\phi}=P_{\phi r}=0\,, \tag{139}$$

where K_1 is the bulk modulus of the inclusion material.

The strain energy density in the inclusion corresponding to the above mentioned stress and strains can be easily evaluated and is

$$\frac{9}{2}K_1(\epsilon-\delta)^2\,.$$

Integrating this over the volume of the sphere of radius a it is seen that the strain energy in the inclusion is W_I given by

$$W_I = 6\pi a^3 K_1 (\epsilon - \delta)^2 . \tag{140}$$

In the case of the matrix, we first of all consider it separately from the inclusion. It is subjected to a uniform radial pressure P_0 at $r = b$. From elementary elasticity theory [6], the radial and hoop stresses for a spherical shell under uniform pressure are of the form

$$p'_{rr} = \frac{A}{r^3} + B , \qquad p'_{\theta\theta} = p'_{\phi\phi} = \frac{-A}{2r^3} + B , \tag{141}$$

and the shear stresses are zero. The constants A and B are determined by the boundary conditions at inner and outer boundaries. At $r = a$, $p'_{rr} = 0$ as the inner boundary is free of external field and at $r = b$, $p'_{rr} = -P_0$. Substituting these boundary conditions in equations (141) we get

$$A = \frac{P_0 a^3 b^3}{(b^3 - a^3)} , \qquad B = \frac{-P_0 b^3}{(b^3 - a^3)} . \tag{142}$$

The displacement field u'_r in the matrix due to P_0 can now be obtained by the relation between stresses and strains, and strains and displacements. It may be shown that the displacement at any point is radial and is equal to

$$u'_r = \frac{-P_0 b^3}{E(b^3 - a^3)} \left\{ \frac{a^3(1+\nu)}{2r^2} + (1-2\nu) r \right\} . \tag{143}$$

Therefore, if $a\epsilon_1$ is the displacement at $r = a$ of the matrix due to the pressure P_0 it is given by $a\epsilon_1 = (u'_r)_{r=a}$, where

$$\epsilon_1 = \frac{-3P_0 b^3(1-\nu)}{2E(b^3-a^3)} . \tag{144}$$

But we observe that the equilibrium boundary is a sphere of radius $a(1+\epsilon)$. Hence the displacement at the inner boundary of the matrix is from a sphere of radius $a(1+\epsilon_1)$ to another of radius $a(1+\epsilon)$. We therefore, have the boundary conditions of the superposing system in terms of the displacement over the inner boundary and stresses at the outer boundary to be given by

$$u_r = a(\epsilon-\epsilon_1) \ \ at \ r = a \ , \ p_{rr} = 0 \ \ at \ r = b . \tag{145}$$

Equations (141) are again utilised to determine the stress field, the constants A and B being replaced by A_1 and B_1. The displacement in terms of these constants is

$$u_r = \frac{1}{E}\left\{\frac{-A_1(1+\nu)}{2r^2} + B_1(1-2\nu)r\right\} . \tag{146}$$

To satisfy the boundary conditions (145) the constants A_1 and B_1 are given by

$$A_1 = \frac{-2a^3b^3(\epsilon-\epsilon_1)E}{b^3(1+\nu)+2a^3(1-2\nu)} , \ B_1 = \frac{2a^3(\epsilon-\epsilon_1)E}{b^3(1+\nu)+2a^3(1-2\nu)} . \tag{147}$$

Thus the stresses and strains everywhere in the matrix, due to the superposing system is known. Equation (40) can be utilised

to determine the strain energy density, which is given by

$$\frac{3}{4E}\left\{\frac{A_1^2(1+\nu)}{r^6}+2B_1^2(1-2\nu)\right\}. \tag{148}$$

Integrating equation (148) over the volume of the matrix and substituting for A_1 and B_1 from equations (147) we get the strain energy in the matrix to be

$$W_m=\frac{4\pi(b^3-a^3)a^3E(\epsilon-\epsilon_1)^2}{b^3(1+\nu)+2a^3(1-2\nu)}. \tag{149}$$

The elastic strain energy of the inclusion and matrix of the system which is disturbing the initial conditions is, therefore, from equations (140) and (149)

$$W=W_I+W_m=6\pi a^3K_1(\epsilon-\delta)^2+\frac{4\pi(b^3-a^3)\,a^3E(\epsilon-\epsilon_1)^2}{b^3(1+\nu)+2a^3(1-2\nu)} \tag{150}$$

Minimising W with respect to ϵ, we obtain that

$$\epsilon=\frac{3K_1\{b^3(1+\nu)+2a^3(1-2\nu)\}\delta+2(b^3-a^3)E\epsilon_1}{3K_1\{b^3(1+\nu)+2a^3(1-2\nu)\}+2(b^3-a^3)E} \tag{151}$$

ϵ_1 in equation (151) is given by equation (144). Putting this value in the value of ϵ, and writing $3K_1(1-2\nu)=E_1$, we get

$$\epsilon=\frac{E_1\{b^3(1+\nu)+2a^3(1-2\nu)\}\delta-3P_0b^3(1-\nu)(1-2\nu)}{E_1\{b^3(1+\nu)+2a^3(1-2\nu)\}+2(b^3-a^3)E(1-2\nu_1)}. \tag{152}$$

The radial and hoop stresses in the matrix in the combined system can be determined by equations (141), the constants being

distinguished as A' and B' which are determined by the following boundary conditions,

$$u_r = a\epsilon \text{ at } r = a\,, \quad p_{rr} = -P_o \text{ at } r = b \tag{153}$$

where ϵ is given by equation (152). Therefore

$$p_{rr} = \frac{A'}{r^3} + B', \quad p_{\theta\theta} = p_{\phi\phi} = \frac{-A'}{2r^3} + B', \tag{154}$$

where

$$A' = \frac{-2a^3b^3\{P_o(1-2\nu) + E\epsilon\}}{\{b^3(1+\nu) + 2a^3(1-2\nu)\}}$$

$$B' = \frac{-P_o b^3(1+\nu) + 2a^3E\epsilon}{\{b^3(1+\nu) + 2a^3(1-2\nu)\}}\,.$$

and the displacement field is,

$$u_r = \frac{1}{E}\left\{\frac{-A'}{2r^2}(1+\nu) + B'(1-2\nu)r\right\}.$$

From equations (152) and (154) the value of the normal stress at $r = a$ is

$$p^{b}_{rr} = \frac{-E_1\{2E\delta(b^3-a^3) + 3P_o b^3(1-\nu)\}}{E_1\{b^3(1+\nu) + 2a^3(1-2\nu)\} + 2E(b^3-a^3)(1-2\nu_1)}\,. \tag{155}$$

The value of P^{b}_{rr} in the inclusion at $r = a$ is, from equations (139) and (152)

$$P^{b}_{rr} = \frac{-E_1\{2E\delta(b^3-a^3) + 3P_o b^3(1-\nu)\}}{E_1\{b^3(1+\nu) + 2a^3(1-2\nu)\} + 2E(b^3-a^3)(1-2\nu_1)}\,. \tag{156}$$

Equations (155) and (156) prove the stress continuity require-

ments.

It may immediately be seen from equations (152) and (156) by setting $\delta = 0$ and $K = K_1$ (where K is the bulk modulus of the matrix material) that the resulting stress-strain field is that when a solid sphere is subjected to a uniform pressure. These stresses and strains can however be derived in the solid sphere by the ordinary classical methods. We shall observe that the results are identical.

CHAPTER XII

Cylindrical Inclusion in Concentric Stressed Finite Matrix

The method of solving inclusion problem explained in the previous chapter, is adopted here to solve a cylindrical inclusion in a concentric stressed matrix under plane strain. Here we consider two cases 1) the matrix under uniform pressure 2) the matrix under uniaxial tension. The results of the second case can be used to obtain solutions of different types of stress field by a little manipulation and or superposition. The abbreviations and notations used here are the same as those in chapter V, for circular inclusion in unstressed matrix. The radii of the initial size of the inclusion is a and of the free surface $a(1+\delta)$. The outer radius of the matrix is b .

We now consider the problem when the pressure at the outer surface of the matrix in the first case is P_0 . We assume the equilibrium boundary to be $a(1+\epsilon)$. As in the case of spherical inclusion, the strains and dilation in the inclusion are

$$\mathcal{E}_{r} = \mathcal{E}_{\theta\theta} = (\epsilon-\delta), \; \mathcal{E}_{r\theta} = 0, \quad \Delta = 2(\epsilon-\delta),$$

whence the stresses $P_{rr}, P_{\theta\theta}, P_{r\theta}$, are

$$P_{rr} = P_{\theta\theta} = 2(\lambda_1 + \mu_1)(\epsilon-\delta), \quad P_{r\theta} = 0 \tag{157}$$

and the strain energy in the inclusion per unit height is

$$W_I = 2(\lambda_1 + \mu_1)(\epsilon-\delta)^2 \pi a^2 . \tag{158}$$

To determine the strain energy in the matrix we have to determine the displacement at its inner boundary due to pressure P_o . The stresses in a circular hollow cylinder due to a uniform normal pressure are of the form [6]

$$p_{rr} = \frac{A}{r^2} + B, \quad p_{\theta\theta} = \frac{-A}{r^2} + B, \quad p_{zz} = 2\nu B, \tag{159}$$

where A and B are constants. These are obtained from the boundary conditions at $r = a$ where $p_{rr} = 0$ and at $r = b$ where $p_{rr} = -P_o$. These are given by

$$A = \frac{P_o a^2 b^2}{(b^2 - a^2)}, \quad B = \frac{-P_o b^2}{(b^2 - a^2)}. \tag{160}$$

Finding the value of $e_{\theta\theta}$ by the strain-stress relation

$$e_{\theta\theta} = \frac{1}{E}\left\{p_{\theta\theta} - \nu\,(p_{rr} + p_{zz})\right\}.$$

and noting that at the inner boundary $e_{\theta\theta} = (u_r/r)_{r=a} = \epsilon_1$ we obtain

$$\epsilon_1 = \frac{-P_o b^2 (1-\nu)}{\mu\,(b^2 - a^2)}. \tag{161}$$

where use has been made of the plane-strain condition that

$$p_{zz} = \nu\,(p_{rr} + p_{\theta\theta})$$

The boundary condition of the superposing system is a mixed one i.e., boundary displacements are prescribed on one part of the surface and the pressure on the other. In the

present case, we have

$$u_r = a(\epsilon - \epsilon_1) \quad \text{at } r = a, \quad p_{rr} = 0 \quad \text{at } r = b \tag{162}$$

The stress field due to these boundary conditions, being perfectly symmetrical, can be evaluated from equations (159), where to distinguish the state of stress and displacement field from that of the free state already defined on page 72, constants A and B have been replaced by A_1 and B_1. The displacement field corresponding to the state of stress given by (159) (and constants A, B replaced by A_1, B_1) will be

$$u_r = \frac{1}{2\mu}\left\{\frac{-A_1}{r} + B_1 r(1-2\nu)\right\} . \tag{163}$$

By the boundary conditions (162) we get the constants

$$A_1 = \frac{-2\mu(\epsilon-\epsilon_1)a^2b^2}{b^2+a^2(1-2\nu)}, \quad B_1 = \frac{2\mu(\epsilon-\epsilon_1)a^2}{\{b^2+a^2(1-2\nu)\}} . \tag{164}$$

The strain energy density of the superposing system in the matrix is

$$\frac{1}{2\mu}\left\{\frac{A_1^2}{r^4} + B_1^2(1-2\nu)\right\} ,$$

which when integrated over the area of the matrix yields

$$W_m = \frac{2\mu(b^2-a^2)(\epsilon-\epsilon_1)^2\pi a^2}{\{b^2+a^2(1-2\nu)\}} , \tag{165}$$

The total strain energy of the system which disturbs the initial configuration is

$$W = W_I + W_m = 2\pi a^2 \left[\frac{\mu(b^2-a^2)(\epsilon-\epsilon_1)^2}{\{b^2+a^2(1-2\nu)\}} + (\lambda_1+\mu_1)(\epsilon-\delta)^2 \right]$$

(166)

Equating $dw/d\epsilon$ to zero and solving for ϵ, we get

$$\epsilon = \frac{(\lambda_1+\mu_1)\{b^2+a^2(1-2\nu)\}\delta + \mu(b^2-a^2)\epsilon_1}{(\lambda_1+\mu_1)\{b^2+a^2(1-2\nu)\} + \mu(b^2-a^2)} .$$

(167)

Substitution for ϵ_1 from equation (161) in equation (167) yields

$$\epsilon = \frac{(\lambda_1+\mu_1)\{b^2+a^2(1-2\nu)\}\delta - P_0 b^2(1-\nu)}{(\lambda_1+\mu_1)\{b^2+a^2(1-2\nu)\} + \mu(b^2-a^2)} .$$

(168)

The stress field in the inclusion is, therefore, obtained by setting this value of ϵ in equations(157) and is

$$P_{rr} = P_{\theta\theta} = \frac{-2(\lambda_1+\mu_1)\{\mu(b^2-a^2)\delta + P_0 b^2(1-\nu)\}}{(\lambda_1+\mu_1)\{b^2+a^2(1-2\nu)\}+\mu(b^2-a^2)} , \quad P_{r\theta} = 0$$

As regards the matrix, the stress field consists of two parts: 1) that obtained by putting the values of A, B from equations (160) in (159) 2) changing the constants A, B in equations (159) by A_1, B_1 and putting the values of

these (A_1, B_1) from equations (164). Of course, we shall express ϵ , ϵ_1 in (164) by their values in equations (168) and (161). The superposition of these fields gives the correct state of stress in the matrix. It can be readily shown by elementary calculations that is continuous at the equilibrium interface.

It may be remarked that the inclusion has undergone a displacement $a\epsilon$ (ϵ given by equation (168)) from its initial size, so also the matrix. Since in the case of the matrix the displacement of its inner boundary in reaching the free state is $a\epsilon_1$ it has further undergone an additional displacement by $a(\epsilon-\epsilon_1)$ due to the reaction of the inclusion. The displacement at the interface is therefore the resultant and is $a\epsilon$, thus proving the continuity of displacement at the interface.

As the second example we consider the case of a circular inclusion in an infinite matrix under uniaxial tension T_0 (say parallel to x-axis). The complex functions $\phi(z)$ and $\psi(z)$ for an infinite plate with a circular hole under the action of T_0 is known [17] . These are

$$\phi(z) = \frac{T_0}{2}\left(\frac{z}{2}+\frac{a^2}{z}\right), \quad \psi(z) = \frac{-T_0}{2}\left\{z + a^2\left(\frac{1}{z}-\frac{a^2}{z^3}\right)\right\} . \tag{169}$$

The only non-vanishing stress at the boundary of the hole is the hoop stress denoted by $p'_{\theta\theta}$ given by

$$p'_{\theta\theta} = T_0(1-2\cos 2\theta) . \tag{170}$$

The displacements at the inner boundary due to T_0 are

$$u'_x = \frac{3T_0(1-\nu)a}{2\mu} = a\epsilon'_1 \ ,$$

$$u'_y = \frac{-T_0(1-\nu)a}{2\mu} = a\epsilon'_2 \ ,$$

(171)

where

$$\epsilon'_1 = \frac{3T_0(1-\nu)}{2\mu} \ , \quad \epsilon'_2 = \frac{-T_0(1-\nu)}{2\mu} \ .$$

(172)

We now assume that the equilibrium boundary of the inclusion and matrix is an ellipse of semi-axes $a(1+\epsilon_1)$ and $a(1+\epsilon_2)$. Here we have introduced two unknown parameters ϵ_1 , ϵ_2 which we shall determine. We shall further show that they satisfy all the requirements at the interface.

The displacement, strains, stresses and strain energy in the inclusion are respectively,

$$U_x = (\epsilon_1 - \delta)x \ , \ U_y = (\epsilon_2 - \delta)y \ , \tag{173}$$

$$\mathcal{E}_{xx} = (\epsilon_1 - \delta) \ , \quad \mathcal{E}_{yy} = (\epsilon_2 - \delta) \ , \quad \mathcal{E}_{xy} = 0 \ ,$$

(174)

$$P_{xx} = \lambda_1(\epsilon_1 + \epsilon_2 - 2\delta) + 2\mu_1(\epsilon_1 - \delta) \ ,$$

$$P_{yy} = \lambda_1(\epsilon_1 + \epsilon_2 - 2\delta) + 2\mu_1(\epsilon_2 - \delta) \ ,$$

$$P_{xy} = 0 \ , \tag{175}$$

$$W_I = \frac{\pi a^2}{2}\left[\lambda_1(\epsilon_1+\epsilon_2-2\delta)^2+2\mu_1\{(\epsilon_1-\delta)^2+(\epsilon_2-\delta)^2\}\right] \tag{176}$$

As regards the matrix, due to the reaction of the inclusion the displacement at the hole is

$$u_x = (\epsilon_1-\epsilon_1')\,x\,, \quad u_y = (\epsilon_2-\epsilon_2')\,y\,, \tag{177}$$

and should entail no stresses at the outer boundary. This is again the second fundamental problem of elasticity theory. Making use of the mapping function $z = a/\zeta$, and following the method described in chapter VI, we get the stress functions

$$F(\zeta) = \frac{A}{k}\,\zeta\,, \quad G(\zeta) = \frac{A}{k}\,\zeta^3 - B\zeta\,,$$

where

$$A = \{(\epsilon-\epsilon_1')-(\epsilon_2-\epsilon_2')\}\frac{a\mu}{2}\,, \quad B = \{(\epsilon_1-\epsilon_1')+(\epsilon_2-\epsilon_2')\}\frac{a\mu}{2} \tag{178}$$

Thus the stresses corresponding to the boundary conditions given in equations (177)

$$p^c_{xx} = \frac{1}{ak}\left\{A(2\cos^2 2\theta-2\cos 2\theta-1)-Bk\cos 2\theta\right\},$$

$$p^c_{yy} = \frac{1}{ak}\left\{A(1-2\cos^2 2\theta-2\cos 2\theta)+Bk\cos 2\theta\right\},$$

$$p^c_{xy} = \frac{1}{ak}\left\{-2A\cos 2\theta\sin 2\theta+Bk\sin 2\theta\right\}.$$

(179)

By calculating the work done at the boundary and relating it to the strain energy, we evaluate the strain energy of the

superposing system in matrix to be

$$W_m = \frac{\mu \pi a^2}{2k}\left\{(k+1)(\epsilon_1-\epsilon_1')^2+(k+1)(\epsilon_2-\epsilon_2')^2+2(k-1)(\epsilon_1-\epsilon_1')(\epsilon_2-\epsilon_2')\right\} \tag{180}$$

Thus the total strain energy changing the free state of the matrix is

$$W = W_I+W_m = \frac{\pi a^2}{2}\Big[\lambda_1(\epsilon_1+\epsilon_2-2\delta)^2+2\mu_1\left\{(\epsilon_1-\delta)^2+(\epsilon_2-\delta)^2\right\}+$$
$$\frac{\mu}{k}\left\{(k+1)(\epsilon_1-\epsilon_1')^2+(k+1)(\epsilon_2-\epsilon_2')^2+2(k-1)(\epsilon_1-\epsilon_1')(\epsilon_2-\epsilon_2')\right\}\Big] \tag{181}$$

For W to be minimum, we obtain from equation (181)

$$\epsilon_1 = \frac{2(\lambda_1+\mu_1)(\mu+\mu_1 k)\delta+\mu\left[\epsilon_1'\{2\mu+\lambda_1+\mu_1(k+1)\}+\epsilon_2'\{\mu_1(k-1)-\lambda_1\}\right]}{2(\lambda_1+\mu_1+\mu)(\mu+\mu_1 k)}$$

$$\epsilon_2 = \frac{2(\lambda_1+\mu_1)(\mu+\mu_1 k)\delta+\mu\left[\epsilon_2'\{2\mu+\lambda_1+\mu_1(k+1)\}+\epsilon_1'\{\mu_1(k-1)-\lambda_1\}\right]}{2(\lambda_1+\mu_1+\mu)(\mu+\mu_1 k)} \tag{182}$$

Putting the values of ϵ_1' and ϵ_2' from equations (172) in equations (182) we obtain

$$\epsilon_1 = \frac{8(\lambda_1+\mu_1)(\mu+\mu_1 k)\delta+T_0(k+1)\{3\mu+\mu_1 k+2(\lambda_1+\mu_1)\}}{8(\lambda_1+\mu_1+\mu)(\mu+\mu_1 k)},$$

$$\epsilon_2 = \frac{8(\lambda_1+\mu_1)(\mu+\mu_1 k)\delta+T_0(k+1)\{\mu_1 k-2(\lambda_1+\mu_1)-\mu\}}{8(\lambda_1+\mu_1+\mu)(\mu+\mu_1 k)} \tag{183}$$

It is now a matter of simple substitution to prove the continuity of the normal and shear stresses and also of the displace-

ments from the initial size of the inclusion.

The solution of the inclusion problem in stressed matrix when the inclusion is spherical or cylindrical, has been obtained by first determining the displacement at the hole of the matrix due to the initial stress field in it. It may be noted that this displacement at the inner boundary need not necessarily be due to an external force. A bonded inclusion in such a medium provides additional complications. For example, if an inclusion is embedded in a matrix having a different coefficient of linear expansion and the system undergoes a change in temperature, the results in this and the previous chapter can be utilised in determining the elastic field. Thus the method suggests an approach to solve boundary value problems of composite sections.

Incidentally we have obtained the solution of a mixed boundary value problem of elasticity theory, when the displacements are given on a part of the boundary and the tractions on the remaining part. The solutions have been obtained, for i) Spherical shell ii) Cylindrical shell which are doubly connected. The solutions in these cases are very simple because of perfect symmetry. Some interesting results which have direct bearing to engineering problems can be derived from the solutions obtained in this and previous chapter. Some of these are given in appendix IV. In the next chapter we consider the more important problem i.e., elliptic inclusion in stressed matrix.

CHAPTER XIII

Elliptic Inclusion in Deforming Matrix - Principal Strain

We shall be concerned with the problem of an inclusion in a matrix deformed by the application of a uniform tensile or compressive stress at infinity, making a constant angle with axis. For this it is necessary to solve two auxiliary problems which we consider in this and the next chapter. In chapter XV we shall combine the results to show how the stress and displacement fields in the inclusion and matrix are affected due to a spontaneous change in the inclusion and an application of an external force field applied to the matrix at infinity.

Consider an infinite elastic medium with an elliptical cavity. Let the boundary of the hole undergo a displacement given by

$$u_x' = \delta_a' x \quad , \qquad u_y' = \delta_b' y \; . \tag{184}$$

No normal and shearing stress are applied at the boundary of the hole. The displacement given by equation (184) would give rise to a stress field in the matrix. In particular there would be a hoop stress at the boundary of the hole.

Consider now that in place of the elliptical cavity, there is an elliptic inclusion tending to undergo a displacement

$$U_x' = \delta_a x \; , \qquad U_y' = \delta_b y \; , \tag{185}$$

in the absence of the outer material, the matrix. This elliptic region, is of a material different from that of the outside material.

We at once observe that if an elliptic region is embedded in the infinite medium with perfect bonds, then the displacements given by equations (184) would not be possible. Because of the presence of the inclusion a stress field in the matrix different from that derived because of boundary displacements given in equations (184) will be obtained. A stress field would be created in the inclusion also. We also observe the reverse case. If the matrix does not tend to undergo the displacements given by (184), but the inclusion tends to undergo the displacements given by (185), a stress field is created in the inclusion and matrix. This latter case has already formed the subject matter of the work done in chapters VII to X. We shall therefore consider the problem formulated in the earlier portion of this paragraph. It might be emphasised, however, that when the matrix undergoes the boundary displacements given in (184) an elastic field is created in the matrix. But when the inclusion undergoes the displacements given by (185), no stresses are developed into it.

The equilibrium boundary is assumed to be an ellipse whose semi-major and semi-minor axes are $a(1+\epsilon_1)$ and $b(1+\epsilon_2)$ respectively, the axes remaining coincident with the initial axes of the hole. Thus in the equilibrium position the boundary displacements of the hole have been taken to be

$$u_x = \epsilon_1 x , \qquad u_y = \epsilon_2 y \ . \tag{186}$$

We have introduced two unknown parameters ϵ_1 , ϵ_2, which have to be determined. For this we apply the methods already discussed

in previous chapters.

We now proceed to evaluate the strain energy in the inclusion. The elastic displacement components of the inclusion boundary from its free surface to the equilibrium boundary are

$$U_x = (\epsilon_1 - \delta_a)\,x \quad , \quad U_y \doteq (\epsilon_2 - \delta_b)\,y \, . \tag{187}$$

For the simply-connected finite region, this also gives the displacement field everywhere in the inclusion. The strains are, therefore,

$$\mathcal{E}_{xx} = (\epsilon_1 - \delta_a) \, , \quad \mathcal{E}_{yy} = (\epsilon_2 - \delta_b) \quad , \quad \mathcal{E}_{xy} = 0 \quad ,$$

and the corresponding stresses are

$$P_{xx} = \lambda_1 (\epsilon_1 + \epsilon_2 - \delta_a - \delta_b) + 2\mu_1 (\epsilon_1 - \delta_a) \, ,$$

$$P_{yy} = \lambda_1 (\epsilon_1 + \epsilon_2 - \delta_a - \delta_b) + 2\mu_1 (\epsilon_2 - \delta_b) \, ,$$

$$P_{xy} = 0 \, . \tag{188}$$

The strain energy in the inclusion will be

$$W_1 = \frac{\pi a b}{2} \left[\lambda_1 (\epsilon_1 + \epsilon_2 - \delta_a - \delta_b)^2 + 2\mu_1 \left\{ (\epsilon_1 - \delta_a)^2 + (\epsilon_2 - \delta_b)^2 \right\} \right] \, . \tag{189}$$

As regards the elliptic boundary of the matrix, it may be noted that it tends to undergo a displacement given by (184), whereby its semi-axes were to be $a(1+\delta_a')$, $b(1+\delta_b')$. In the final equilibrium position the boundary is the ellipse of semi-axes $a(1+\epsilon_1)$, $b(1+\epsilon_2)$. Thus the displacement components

of the boundary of the hole are

$$u_x = (\epsilon_1 - \delta_a')\, x = \frac{\epsilon_1 - \delta_a'}{2}\,(z + \bar{z})\,,$$

$$u_y = (\epsilon_2 - \delta_b')\, y = \frac{\epsilon_2 - \delta_b'}{2i}\,(z - \bar{z})\,. \tag{190}$$

We may remark that this is again the second fundamental boundary value problem of the classical elasticity theory. Using the complex variable techniques already enunciated in chapter VI and exemplified in chapters VII and IX, and further noting that we should use the displacemnt parameters $(\epsilon_1 - \delta_a')$ and $(\epsilon_2 - \delta_b')$ in place of ϵ_1 and ϵ_2 in equations (102), we obtain the complex potential functions $F(\zeta)$ and $G(\zeta)$. These have the same meaning as given in chapter VI. From the results in chapter VII, we obtain

$$F(\zeta) = \frac{A\zeta}{k}\,, \quad G(\zeta) = \frac{\zeta(\zeta^2 + m)}{(1 - m\zeta^2)}\,\frac{A}{k} - B\zeta\,, \tag{191}$$

where

$$A = \mu\{a(\epsilon_1 - \delta_a') - b(\epsilon_2 - \delta_b')\}\,, \quad B = \mu\{a(\epsilon_1 - \delta_a') + b(\epsilon_2 - \delta_b')\} \tag{192}$$

These functions determine the elastic field everywhere. The stresses at the boundary of the hole will be the same as in equations (108), except that the constants A and B will have the values given in equations (192). These stresses are,

$$\overset{c}{p}_{xx} = \frac{A}{kR}\,\frac{\{(4m+2)\cos^2 2\theta - (m^3+6m^2+3m+2)\cos 2\theta + (2m^3+3m^2+2m-1)\}}{(m^2-2m\cos 2\theta+1)^2}$$

$$+\frac{B}{R}\,\frac{(m-\cos 2\theta)}{(m^2-2m\cdot\cos 2\theta+1)},$$

$$\overset{c}{p}_{yy} = \frac{A}{kR}\,\frac{\{(4m-2)\cos^2 2\theta + (m^3-6m^2+3m-2)\cos 2\theta + (2m^3-3m^2+2m+1)\}}{(m^2-2m\cos 2\theta+1)^2}$$

$$-\frac{B}{R}\,\frac{(m-\cos 2\theta)}{(m^2-2m\cos 2\theta+1)},$$

$$\overset{c}{p}_{xy} = \frac{A}{kR}\,\frac{(3m-m^2-2\cos 2\theta)\sin 2\theta}{(m^2-2m\cos 2\theta+1)^2} + \frac{B}{R}\,\frac{\sin 2\theta}{(m^2-2m\cos 2\theta+1)}.$$

(193)

Using these values of stresses and boundary displacements the strain energy of the disturbing system can be calculated by determining the work done as in chapter VII and relating it to the strain energy. After some simplifications, we obtain the strain energy W_m in the matrix to be

$$W_m = \frac{\mu\pi ab}{2k}\Big[(k+1)\Big\{\frac{a}{b}(\epsilon_1-\delta_a')^2 + \frac{b}{a}(\epsilon_2-\delta_b')^2\Big\} + 2(k-1)(\epsilon_1-\delta_a')(\epsilon_2-\delta_b')\Big].$$

(194)

The total strain energy of the superposing system is, from equations (189) and (194).

$$W = W_I + W_m = \frac{\pi ab}{2}\Big[\lambda_1(\epsilon_1+\epsilon_2-\delta_a-\delta_b)^2 + 2\mu_1\{(\epsilon_1-\delta_a)^2+(\epsilon_2-\delta_b)^2\}\Big] +$$

$$\frac{\mu\pi ab}{2k}\Big[\frac{(k+1)}{ab}\{a^2(\epsilon_1-\delta_a')^2 + b^2(\epsilon_2-\delta_b')^2 + 2(k-1)(\epsilon_1-\delta_a')(\epsilon_2-\delta_b')\Big].$$

(195)

Equating the partial derivatives of W with respect to ϵ_1 and ϵ_2 to zero gives the following two equations.

$$\frac{\mu}{k}\left\{\frac{a}{b}(\epsilon_1-\delta_a')(k+1)+(\epsilon_2-\delta_b')(k-1)\right\}+\mu_1\left\{\frac{\lambda_1}{\mu_1}(\epsilon_1+\epsilon_2-\delta_a-\delta_b)+2(\epsilon_1-\delta_a)\right\}=0$$

$$\frac{\mu}{k}\left\{\frac{b}{a}(\epsilon_2-\delta_b')(k+1)+(\epsilon_1-\delta_a')(k-1)\right\}+\mu_1\left\{\frac{\lambda_1}{\mu_1}(\epsilon_1+\epsilon_2-\delta_a-\delta_b)+2(\epsilon_2-\delta_b)\right\}=0 \tag{196}$$

Solving for ϵ_1 and ϵ_2 yields

$$\epsilon_1=\frac{\begin{aligned}\Big[&\{b\mu(k+1)(\lambda_1+2\mu_1)+4\mu_1k(\lambda_1+\mu_1)a-\lambda_1\mu a(k-1)\}b\delta_a\\&+\mu\{\lambda_1(k+1)b-(\lambda_1+2\mu_1)(k-1)a\}b\delta_b+a\mu\{4b\mu-\lambda_1b(k-1)\\&+a(k+1)(\lambda_1+2\mu_1)\}\delta_a'+b\mu\{a(k-1)(\lambda_1+2\mu_1)-b\lambda_1(k+1)\}\delta_b'\Big]\end{aligned}}{4ab\{\mu^2+\mu_1k(\lambda_1+\mu_1)\}+\lambda_1\mu\{(a+b)^2+(a-b)^2k\}+2\mu\mu_1(k+1)(a^2+b^2)}$$

$$\epsilon_2=\frac{\begin{aligned}\Big[&\{a\mu(k+1)(\lambda_1+2\mu_1)+4\mu_1k(\lambda_1+\mu_1)b-\lambda_1\mu b(k-1)\}a\delta_b\\&+\mu\{\lambda_1(k+1)a-(\lambda_1+2\mu_1)(k-1)b\}a\delta_a+b\mu\{4a\mu-\\&\lambda_1a(k-1)+b(k+1)(\lambda_1+2\mu_1)\}\delta_b'+a\mu\{b(k-1)(\lambda_1+2\mu_1)\\&-a\lambda_1(k+1)\}\delta_a'\Big]\end{aligned}}{\begin{aligned}&4ab\{\mu^2+\mu_1k(\lambda_1+\mu_1)\}+\lambda_1\mu\{(a+b)^2+(a-b)^2k\}+\\&2\mu\mu_1(k+1)(a^2+b^2)\end{aligned}} \tag{197}$$

These results should satisfy the condition of the continuity of the normal and shear stresses at the interface. These stresses with the notations used are given by,

$$P^b_{nn} = p^b_{nn} = \frac{-1}{2ab}\left[\frac{A(a-b)}{k} + B(a+b)\right] - \left[\frac{A(a+b)}{k} + B(a-b)\right]\frac{\cos 2\beta}{2ab},$$

$$P^b_{ns} = p^b_{ns} = \left[\frac{A(a+b)}{k} + B(a-b)\right]\frac{\sin 2\beta}{2ab}$$

(198)

where the constants A and B are determined by substituting the values of ϵ_1 and ϵ_2 from equations (197) in equations (192). These are given by

$$A = \frac{\begin{array}{l} 2\mu abk\left[\{2a\mu_1(\lambda_1+\mu_1) - \lambda_1\mu(a-b) + 2\mu\mu_1 b\}(\delta_a - \delta_a') \right. \\ \left. \quad - \{2b\mu_1(\lambda_1+\mu_1) + \lambda_1\mu(a-b) + 2\mu\mu_1 a\}(\delta_b - \delta_b')\right] \end{array}}{4ab\{\mu_1^2 + \mu_1 k(\lambda_1+\mu_1)\} + \lambda_1\mu\{(a+b)^2 + (a-b)^2 k\} + 2\mu\mu_1(k+1)(a^2+b^2)},$$

$$B = \frac{\begin{array}{l} 2\mu ab\left[\{2a\mu_1(\lambda_1+\mu_1)k + \lambda_1\mu(a+b) + 2\mu\mu_1 b\}(\delta_a - \delta_a') \right. \\ \left. \quad + \{2b\mu_1(\lambda_1+\mu_1)k + \lambda_1\mu(a+b) + 2\mu\mu_1 a\}(\delta_b - \delta_b')\right] \end{array}}{4ab\{\mu_1^2 + \mu_1 k(\lambda_1+\mu_1)\} + \lambda_1\mu\{(a+b)^2 + (a-b)^2 k\} + 2\mu\mu_1(k+1)(a^2+b^2)}.$$

(199)

The hoop stresses in the inclusion and matrix due to the effect of the deformation of the inclusion are

$$P^b_{ss} = \frac{-1}{2ab}\left[\frac{A(a-b)}{k} + B(a+b)\right] + \left[\frac{A(a+b)}{k} + B(a-b)\right]\frac{\cos 2\beta}{2ab},$$

$$p^b_{ss} = \frac{-1}{2ab}\left[\frac{3A(a-b)}{k} - B(a+b)\right] - \left[\frac{3A(a+b)}{k} - B(a-b)\right]\frac{\cos 2\beta}{2ab}.$$

(200)

It should be noted, however, that the resultant hoop stress at the inner boundary of the matrix is the sum of p_{ss}^{σ} given in equations (200) and the initial hoop stress referred to in paragraph 2 of this chapter.

CHAPTER XIV

Elliptic Inclusion in Deforming Matrix - Shear Strain

Consider an elliptic hole in an infinite medium, the boundary of which undergoes a displacement

$$u'_x = \gamma'_a y , \qquad u'_y = \gamma'_b x . \tag{201}$$

This displacement gives rise to a stress field in the matrix. Let in the initial undeformed elliptic hole an inclusion be embedded, which in the absence of the matrix would undergo a non-elastic displacement

$$U'_x = \gamma_a y , \qquad U'_y = \gamma_b x . \tag{202}$$

Because of the mutual constraints, the initial elastic field in the matrix is changed and an elastic field is created in the inclusion also. Further, the equilibrium boundary will be different from those given by (201) or (202). We determine these in this chapter.

Let us assume that the equilibrium boundary is obtained by giving a displacement

$$u_x = \gamma_1 y , \qquad u_y = \gamma_2 x . \tag{203}$$

to the inner boundary of matrix. The boundary elastic displacement of the inclusion from its free surface to the equilibrium surface is given by

$$U_x = (\gamma_1 - \gamma_a)\, y \quad , \quad U_y = (\gamma_2 - \gamma_b)\, x \tag{204}$$

By the fact that the inclusion is simply connected and finite

where the constants A_1 and B_1 are now given by

$$A_1 = \mu\{b(\gamma_1-\gamma_a') + a(\gamma_2-\gamma_b')\}, \quad B_1 = \mu\{a(\gamma_2-\gamma_b') - b(\gamma_1-\gamma_a')\}. \tag{209}$$

Having obtained the complex potentials $F(\zeta)$ and $G(\zeta)$ it is simple to evaluate the stress field everywhere. In particular the stresses at the boundary of the hole are

$$p_{xx}^c = \frac{A_1\{(2+3m+2m^2-m^3) - 2(2m+1)\cos 2\theta\}\sin 2\theta}{kR(m^2-2m\cos 2\theta+1)^2} - \frac{B_1 \sin 2\theta}{R(m^2-2m\cos 2\theta+1)},$$

$$p_{yy}^c = \frac{A_1\{(m^3+2m^2-3m+2) - 2(2m-1)\cos 2\theta\}\sin 2\theta}{kR(m^2-2m\cos 2\theta+1)^2} + \frac{B_1 \sin 2\theta}{R(m^2-2m\cos 2\theta+1)},$$

$$p_{xy}^c = \frac{-A_1\{2\cos^2 2\theta - (3m+m^3)\cos 2\theta + (3m^2-1)}{kR(m^2-2m\cos 2\theta+1)^2} + \frac{B_1(m-\cos 2\theta)}{R(m^2-2m\cos 2\theta+1)}. \tag{210}$$

Using the Clapeyron's theorem, it can be proved by the method followed in chapter IX, that the strain energy is

$$W_m = \frac{\mu \pi a b}{2k}\Big\{(k+1)\frac{b}{a}(\gamma_1-\gamma_a')^2 + 2(1-k)(\gamma_1-\gamma_a')(\gamma_2-\gamma_b') + (k+1)\frac{a}{b}(\gamma_2-\gamma_b')^2\Big\} \tag{211}$$

The total strain energy in the matrix and the inclusion due to the spontaneous deformation of the inclusion is given by

$$W = W_I + W_m = \frac{\pi ab}{2}\left[\frac{\mu(k+1)}{kab}\left\{b^2(\gamma_1-\gamma_a')^2 + a^2(\gamma_2-\gamma_b')^2\right\}\right.$$

$$\left. + \frac{2(1-k)\mu}{k}(\gamma_1-\gamma_a')(\gamma_2-\gamma_b') + \mu_1(\gamma_1+\gamma_2-\gamma_a-\gamma_b)^2\right] . \qquad (212)$$

Minimising W with respect to γ_1 and γ_2 we obtain

$$\gamma_1 = \frac{\mu_1 a\{(a-b)+(a+b)k\}(\gamma_a+\gamma_b-\gamma_b') + \gamma_a' b[4a\mu+\mu_1\{(a+b)k-(a-b)\}]}{4\mu ab+\mu_1\{(a-b)^2+k(a+b)^2\}},$$

$$\gamma_2 = \frac{\mu_1 b\{(a+b)k-(a-b)\}(\gamma_a+\gamma_b-\gamma_a') + \gamma_b' a[4b\mu+\mu_1\{(a+b)k+(a-b)\}]}{4\mu ab+\mu_1\{(a-b)^2+k(a+b)^2\}}. \qquad (213)$$

The values of the constants A_1 , B_1 , in the complex functions of equations (208) can now be obtained in terms of the known quantities by substituting the values of γ_1 , γ_2 from equations (213) in (209).

We have still to prove that this satisfies the continuity requirements of the normal and shear stresses at the interface, and also of the displacements. The normal and shear stresses in the inclusion are obtained by first putting the values of γ_1, γ_2 in equations (205) and then using the transformation formule (117). Similarly for the matrix, these would be obtained by first putting the values of A_1 , B_1 in terms of known quantities in equations (208) and then making use of the transformation formule (117) and the trigonomatrical relations (119). It will be observed that the normal and shear stresses are continuous at the boundary and are given by

$$P_{mn}^{\iota} = p_{mn}^{\iota} = \frac{-4\mu\mu_1(\gamma_a+\gamma_b-\gamma_a'-\gamma_b')\,ab\,\sin 2\beta}{4ab\mu+\mu_1\{(a-b)^2+(a+b)^2k\}},$$

$$P_{ns}^{b} = p_{ns}^{b} = -\frac{4\mu\mu_1(\gamma_a+\gamma_b-\gamma_a'-\gamma_b')\, ab\cos 2\beta}{4ab\mu+\mu_1\{(a-b)^2+(a+b)^2k\}} \, . \tag{214}$$

The hoop stress at the equilibrium boundary of the inclusion is easily seen to be P_{ss}^{b} given by

$$P_{ss}^{b} = -P_{nn}^{b}$$

from the invariance condition of the hydrostatic stress field $P_{xx}+P_{yy} = P_{nn}+P_{ss}$ (= 0 in this case). But the hoop stress at the equilibrium boundary of the matrix is to be determined by the superposition of its value due to the initial displacement given by equations (201) on the field due to the non-elastic deformation of the inclusion.

It may easily be seen that all the previous calculations are based on the fact that there is a equilibrium interface and hence the displacements are continuous. However, it may be noted that the resultant displacement of the interface should be measured from the initial size of the hole.

In this and the previous chapter, we have considered two separate problems.

As a first problem we consider an elliptic region in an infinite medium, tending to undergo non-elastic displacement

$$U_x' = \delta_a x + \gamma_a y \, , \qquad U_y' = \delta_b y + \gamma_b x \, ,$$

in the absence of the matrix. Due to the constraints of the matrix, this would generate a system of locked up accommodation stresses both in the inclusion and its surrounding material.

As a next problem, consider the elliptic boundary in the absence of the inclusion in the infinite medium tending to undergo a displacement given by

$$u'_x = \delta'_a x + \gamma'_a y \ , \qquad u'_y = \delta'_b y + \gamma'_b x \ .$$

But due to the constraints of the material in the cavity it would not be able to attain its free state. A system of stress field both in itself and in the inclusion would be generated.

The first problem has been considered in chapter X. The second problem can be considered by taking δ_a , δ_b, γ_a , γ_b to be zero in the results of this and the previous chapter. In the general case when the problem referred to in the previous two paragraphs are combined, i.e., when both the inclusion and elliptic boundary of the matrix tend to undergo a change stated earlier, the equilibrium interface is obtained by giving a displacement whose components are

$$u_x = \epsilon_1 x + \gamma_1 y \ , \qquad u_y = \epsilon_2 y + \gamma_2 x \ ,$$

to the boundary of the hole. The values of ϵ_1 , ϵ_2 , γ_1 , γ_2 are given by equations (197) and (213).

CHAPTER XV

Elliptic Inclusion in a Large Plate Under Tension and Related Problems

We now consider the more interesting and technologically important problem. What happens if an inclusion embedded in the matrix undergoes a non-elastic deformation and further the matrix is subjected to a uniform tension T at infinity inclined at an angle α to the major axis? How is the elastic field or the equilibrium boundary disturbed both in the inclusion and the matrix? We shall see that the results obtained in the previous chapters can be utilised to obtain explicit answers to these questions.

First we consider the stress field in the matrix containing an elliptic cavity in an infinite medium under tension T . The solution of this problem is known in [8,17] among others. Using the notations of previous chapters, the functions $F(\zeta)$ and $G(\zeta)$ are given by

$$F(\zeta) = \frac{TR\zeta}{4}\left(2e^{2i\alpha} - m + \frac{1}{\zeta^2}\right),$$

$$G(\zeta) = \frac{-TR}{2}\left\{\frac{e^{-2i\alpha}}{\zeta} + \frac{e^{2i\alpha}\zeta}{m} - \frac{(1+m^2)(e^{2i\alpha}-m)\zeta}{m(1-m\zeta^2)}\right\}. \tag{215}$$

From equations (229) and (86) the displacement components u_x', u_y' at the inner boundary of the matrix, are given by

$$2\mu(u_x' + iu_y') = \frac{k+1}{4}\,TR\left\{2e^{i(2\alpha-\theta)} - me^{-i\theta} + e^{i\theta}\right\}, \tag{216}$$

where ζ has been taken to be $e^{i\theta}$ at the boundary. By separating the real and imaginary parts, and noting that

$$x = a \cos\theta, \qquad y = -b \sin\theta,$$

we obtain

$$u'_x = \frac{T(k+1)}{8\mu ab}\Big[\{(a+b)\cos 2\alpha + b\}bx + (a+b)ay \sin 2\alpha\Big],$$

$$u'_y = \frac{T(k+1)}{8\mu ab}\Big[\{a-(a+b)\cos 2\alpha\}ay + b(a+b)x \sin 2\alpha\Big],$$

(217)

which may be written as

$$u'_x = \delta'_a x + \gamma'_a y, \qquad u'_y = \delta'_b y + \gamma'_b x, \tag{218}$$

where

$$\delta'_a = \frac{T(k+1)}{8\mu a}\{b + (a+b)\cos 2\alpha\},$$

$$\delta'_b = \frac{T(k+1)}{8\mu b}\{a - (a+b)\cos 2\alpha\},$$

$$\gamma'_a = \frac{T(k+1)}{8\mu b}(a+b)\sin 2\alpha, \quad \gamma'_b = \frac{T(k+1)}{8\mu a}(a+b)\sin 2\alpha.$$

(219)

The values of $\delta'_a, \delta'_b, \gamma'_a, \gamma'_b$ in equations (219) indicate that they can be simultaneously zero only when $T = 0$. Further γ'_a, γ'_b will be zero when $\alpha = 0$ or $\pi/2$ which is obvious on physical grounds. Both of δ'_a and δ'_b cannot be zero simultaneously for a non-zero value of T.

If further the inclusion undergoes a non-elastic displacement given by

$$U'_x = \delta_a x + \gamma_a y\,, \qquad U'_y = \delta_b y + \gamma_b x\,, \tag{220}$$

then the equilibrium boundary will be obtained by giving a displacement

$$u_x = \epsilon_1 x + \gamma_1 y\,, \qquad u_y = \epsilon_2 y + \gamma_2 x\,, \tag{221}$$

to the interface as shown in previous chapters. The values of ϵ_1, ϵ_2, γ_1, γ_2 from equations (197) and (213) shall be

$$\epsilon_1 = \frac{\begin{aligned}\Big[&32\,ab\,\mu_1 k(\lambda_1+\mu_1)\delta_a + 8\lambda_1\mu b\{(a+b)-(a-b)k\}(\delta_a+\delta_b) \\ &+ 16\mu\mu_1\{(k+1)b\delta_a-(k-1)a\delta_b\}b + T(k+1)\{4b^2\mu \\ &+ \lambda_1(a^2-b^2)(k-1)+2\mu_1 a\{(a+b)k-(a-b)\} \\ &+ 2(a+b)\cos 2\alpha\{2b\mu+\lambda_1(a+b)+2a\mu_1\}\}\Big]\end{aligned}}{\begin{aligned}8\Big[&4ab\{\mu^2+k\mu_1(\lambda+\mu)\}+\lambda_1\mu\{(a+b)^2+(a-b)^2k\}+ \\ &2\mu\mu_1(k+1)(a^2+b^2)\Big]\end{aligned}}\,,$$

$$\epsilon_2 = \frac{\begin{aligned}\Big[&32\,ab\,\mu_1 k(\lambda_1+\mu_1)\delta_b + 8\lambda_1\mu a\{(a+b)+(a-b)k\}(\delta_a+\delta_b) \\ &+ 16\mu\mu_1 a\{(k+1)a\delta_b-(k-1)b\delta_a\}+T(k+1)\{4a^2\mu \\ &- \lambda_1(a^2-b^2)(k-1)+2\mu_1 b\{(a+b)k+(a-b)\} \\ &- 2(a+b)\cos 2\alpha\{2a\mu+\lambda_1(a+b)+2b\mu_1\}\}\Big]\end{aligned}}{\begin{aligned}8\Big[&4ab\{\mu^2+k\mu_1(\lambda_1+\mu)\}+\lambda_1\mu\{(a+b)^2+(a-b)^2k\}+ \\ &2\mu\mu_1(k+1)(a^2+b^2)\Big]\end{aligned}}\,,$$

$$\gamma_1 = \frac{4\mu\mu_1 a\{(a+b)k+(a-b)\}(\gamma_a+\gamma_b)+T(k+1)(a+b)\{2a\mu-\mu_1(a-b)\operatorname{Sin}2\alpha\}}{4\mu[4\mu ab+\mu_1\{(a-b)^2+k(a+b)^2\}]},$$

$$\gamma_2 = \frac{4\mu\mu_1 b\{(a+b)k-(a-b)\}(\gamma_a+\gamma_b)+T(k+1)(a+b)\{2b\mu+(a-b)\mu_1\operatorname{Sin}2\alpha\}}{4\mu(4\mu ab+\mu_1\{(a-b)^2+k(a+b)^2\}]},$$

(222)

where we have put the values of δ_a', δ_b', γ_a', γ_b', from equations (219). Substituting equations (222) in (192) and (209) the constants A, B, A_1, B_1 are found to be

$$A = \frac{\begin{array}{c}[8\mu abk\{2\mu_1(\lambda_1+\mu_1)(a\delta_a-b\delta_b)-\lambda_1\mu(a-b)(\delta_a+\delta_b)+2\mu\mu_1(b\delta_a-a\delta_b)\} \\ +T(k+1)k(a+b)\{\mu(\lambda_1+2\mu_1)\{(a-b)^2-(a^2-b^2)\operatorname{Cos}2\alpha\}+2\mu_1 ab(\lambda_1+\mu_1+\mu)\}]\end{array}}{4[4ab\{\mu^2+k\mu_1(\lambda_1+\mu_1)\}+\lambda_1\mu\{(a+b)^2+(a-b)^2k\}+2\mu\mu_1(k+1)(a^2+b^2)]},$$

$$B = \frac{\begin{array}{c}[8\mu ab\{2\mu_1(\lambda_1+\mu_1)k(a\delta_a+b\delta_b)+\lambda_1\mu(a+b)(\delta_a+\delta_b)+2\mu\mu_1(b\delta_a+a\delta_b)\} \\ -T(k+1)(a+b)\{2\mu_1 ab\{(\lambda_1+\mu_1)k-\mu\}+\mu(\lambda_1+2\mu_1)\{(a^2+b^2)-(a^2-b^2) \\ \operatorname{Cos}2\alpha\}\}]\end{array}}{4[4ab\{\mu^2+\mu_1 k(\lambda_1+\mu_1)\}+\lambda_1\mu\{(a+b)^2+(a-b)^2k\}+2\mu\mu_1(k+1)(a^2+b^2)},$$

$$A_1 = \frac{\mu_1(a+b)k\{8(\gamma_a+\gamma_b)\mu ab-T(k+1)(a+b)^2\operatorname{Sin}2\alpha\}}{4[4\mu ab+\mu_1\{(a-b)^2+(a+b)^2k\}]},$$

$$B_1 = \frac{-\mu_1(a-b)\{8(\gamma_a+\gamma_b)\mu ab-T(k+1)(a+b)\operatorname{Sin}2\alpha\}}{4[4\mu ab+\mu_1\{(a-b)^2+(a+b)^2k\}]}.$$

(223)

The normal and shear stresses at the common boundary in terms of

these constants given by equations (223) are

$$P_{nn}^{b} = p_{nn}^{b} = -\frac{1}{2abk}\Big[\{A(a-b)+Bk(a+b\}+\{A(a+b)+Bk(a-b)\}$$
$$\cos 2\beta+\{A_1(a+b)+B_1k(a-b)\}\sin 2\beta\Big],$$

$$P_{ns}^{b} = p_{ns}^{b} = \frac{1}{2abk}\Big[\{A(a+b)+Bk(a-b)\}\sin 2\beta-\{A_1(a+b)$$
$$+B_1k(a-b)\}\cos 2\beta\Big]. \tag{224}$$

The hoop stress in the inclusion at its boundary is

$$P_{ss}^{b} = -\frac{1}{2abk}\Big[\{A(a-b)+Bk(a+b)\}-\{A(a+b)+Bk(a-b)\}$$
$$\cos 2\beta-\{A_1(a+b)+B_1k(a-b)\}\sin 2\beta\Big], \tag{225}$$

and in the matrix its value due to the deformation of inclusion is

$$p_{ss}^{b} = -\frac{1}{2abk}\Big[\{3A(a-b)-Bk(a+b)\}+\{3A(a+b)-Bk(a-b)\}$$
$$\cos 2\beta+\{3A_1(a+b)-B_1k(a-b)\}\sin 2\beta \tag{226}$$

It may be noted that ϵ_1, ϵ_2 as well as γ_1, γ_2 will not be zero, even when δ_a, δ_b, γ_a, γ_b are zero. This refers to the case of an elliptic inhomogeneity in an infinite medium subjected to a uniform tension inclined at an angle α to the major axis. If further the elastic properties of the inclusion are the same as those of the outside material, then we recover the displacements at the boundary of the elliptic region due to the tension. We can similarly solve the Eshelby's transformation problem just by putting λ_1, μ_1 equal to λ, μ. It is interesting to observe that even when γ_a and γ_b are equal to zero, γ_1 and γ_2 are not zero for the given tension. These values can be zero only when $\alpha = 0$ or $\pi/2$.

As a particular case if we set λ_1, μ_1 equal to zero, which means there is no inclusion, we at once recover the results

$$\epsilon_1 = \delta_a' = \frac{T(k+1)}{8\mu a}\left\{b+(a+b)\cos 2\alpha\right\}, \quad \epsilon_2 = \delta_b' = \frac{T(k+1)}{8\mu b}\left\{a-(a+b)\cos 2\alpha\right\},$$

$$\gamma_1 = \gamma_a' = \frac{T(k+1)}{8\mu b}(a+b)\sin 2\alpha, \quad \gamma_2 = \gamma_b' = \frac{T(k+1)}{8\mu a}(a+b)\sin 2\alpha,$$

which again is a check on the result. It may also be observed that if $a=b$

$$\delta_a' = \frac{T(k+1)(1+2\cos 2\alpha)}{8\mu}, \quad \delta_b' = \frac{T(k+1)(1-2\cos 2\alpha)}{8\mu},$$

$$\gamma_a' = \gamma_b' = \frac{T(k+1)\sin 2\alpha}{4\mu}.$$

On the other hand if the length of the major axis is large as compared to that of the minor axis, δ_b' would be larger as compared to δ_a' except when α is very nearly zero. This is otherwise also true on physical considerations.

Similarly in the presence of the inclusion, the corresponding results for circular case are obtained by setting $a=b$ in the relevant equations. For the case of a slender inclusion the results are obtained by a limiting process. For this case the values of ϵ_1, ϵ_2, γ_1 and γ_2 are given by

$$\epsilon_1 = \frac{T}{8\mu}\left(k-1+2\cos 2\alpha\right),$$

$$\epsilon_2 = \delta_b + \frac{\lambda_1\delta_a}{\lambda_1+2\mu_1} + \frac{T\{4\mu(1-\cos 2\alpha)-\lambda_1(k-1+2\cos 2\alpha)\}}{8(\lambda_1+2\mu_1)\mu},$$

$$\gamma_1 = (\gamma_a+\gamma_b) + \frac{T(2\mu-\mu_1)\sin 2\alpha}{4\mu\mu_1}, \quad \gamma_2 = \frac{T}{4\mu}\sin 2\alpha .$$

Of course, we recover all the results of chapters VII to X by setting T equal to zero.

The analysis enables us to solve another class of problems. These relate to the case when an infinite material containing an elliptic hole under uniform tension at infinity, is subjected to some special type of loading conditions at the inner boundary.

For example if in equations (224)

$$A(a+b) + B(a-b)k = 0,$$

$$A_1(a+b) + B_1(a-b)k = 0,$$

$$\frac{1}{2abk}\left\{A(a-b) + Bk(a+b)\right\} = p,$$

we shall have three linear simultaneous equations to determine the corresponding values of δ_a, δ_b and $(\gamma_a+\gamma_b)$ in terms of p and T. This would solve the case of a matrix subjected to a tension at infinity and uniform pressure at its inner boundary. These results check up with the solution of the problem if solved by alternative classical methods.

Many other particular boundary value problems relating to

a large plate with an elliptic hole can similarly be solved by properly choosing the constants in equations (224) or in (193) or in (210). The latter set of equations can deal with some cases when the inner boundary is subjected to a particular type of boundary tractions, for example, results of a pure shear at infinity can be obtained by superposition of two equal and opposite tensions at right angles to each other.

The results obtained in previous chapters are for those of isotropic elastic medium. A whole field of many other interesting cases can be thought of. For example, what happens when a material containing an inclusion is subjected to bending or twisting and so on.

From the technological point of view, it is more interesting to consider the anisotropic materials. The next in order of simplicity are cubic and orthotropic materials. We shall deal with such elastic cases. Unfortunately not many problems related to anisotropic medium are available. Hence the solution have to be built up from the beginning. We shall very briefly review the theory in the next chapter.

PART II

CHAPTER XVI

Anisotropic Elasticity

The stress-equilibrium equations (6) depend upon the static nature of an element of volume and forces acting on it. These do not depend upon the properties of the material. So also are the strain displacement relations (8) and the compatibility equations (13) which depend upon the continuity requirements. Similarly the boundary conditions (14) or (15) depend upon the specified values of tractions or displacements as the case may be. Hence the stress-equilibrium equations, the strain-displacement relations, compatibility conditions and boundary conditions would remain unaltered even in anisotropic medium.

The fundamental difference comes in the constants relating the stresses to strains and vice-versa. These relations are expressed by the generalised Hooke's Law as stated in equation (7). Hence if p_{ij} and e_{ij} be the stress and strain components, then

$$e_{ij} = c_{ijkl} p_{kl} ,$$

where c_{ijkl} are invariants so far as the stresses and strains are concerned but change with the transformation of the system of coordinates. Although apparently there are 81 constants, due to symmetry and energy considerations the number of constants reduce to 21 in a completely anisotropic medium.

The number of constants further reduce if the body has some plane or planes of symmetry, or axis or axes of symmetry. If the figure allows a reflexion in a plane it is said to possess

a plane of symmetry. Similarly a figure which allows a rotation about an axis is said to have an axis of symmetry. Of course, every operation which is neither a rotation about an axis, nor a reflexion in a plane, may be proved to be the combination of such operations. As this discussion is outside the scope of this thesis, we shall not be concerned with it.

In cartesian notations the strain-stress relations can be simply put in the following form.

$$
\begin{aligned}
e_{xx} &= C_{11} p_{xx} + C_{12} p_{yy} + C_{13} p_{zz} + C_{14} p_{yz} + C_{15} p_{zx} + C_{16} p_{xy} , \\
e_{yy} &= C_{21} p_{xx} + C_{22} p_{yy} + C_{23} p_{zz} + C_{24} p_{yz} + \cdots + \cdots , \\
e_{zz} &= \cdots + \cdots + \cdots + \cdots + \cdots + \cdots , \\
2e_{yz} &= \cdots + \cdots + \cdots + \cdots + \cdots + \cdots , \\
2e_{zx} &= \cdots + \cdots + \cdots + \cdots + \cdots + \cdots , \\
2e_{xy} &= C_{61} p_{xx} + C_{62} p_{yy} + C_{63} p_{zz} + C_{64} p_{yz} + C_{65} p_{zx} + C_{66} p_{xy} .
\end{aligned}
\tag{227}
$$

Note that in this the constants C_{ij} may be related with the coefficients C_{ijkl} in (7). Further it may be seen that in these relations $C_{ij} = C_{ji}$ since the net work done on loading and subsequent unloading of the material should be zero, regardless of the order of application or removal of the various components of stress. C_{ij}'s are usually called elastic constants or moduli.

Similarly the stresses can be put in terms of strains by inverting the equations (227). These can be written as

$$
\begin{aligned}
p_{xx} &= S_{11} e_{xx} + S_{12} e_{yy} + S_{13} e_{zz} + 2 S_{14} e_{yz} + 2 S_{15} e_{zx} + 2 S_{16} e_{xy} ,\\
p_{yy} &= S_{21} e_{xx} + S_{22} e_{yy} + \cdots + \cdots + \cdots + \cdots ,\\
p_{zz} &= \cdots + \cdots + \cdots + \cdots + \cdots + \cdots ,\\
p_{yz} &= \cdots + \cdots + \cdots + \cdots + \cdots + \cdots ,\\
p_{zx} &= \cdots + \cdots + \cdots + \cdots + \cdots + \cdots ,\\
p_{xy} &= S_{61} e_{xx} + S_{62} e_{yy} + S_{63} e_{zz} + 2 S_{64} e_{yz} + 2 S_{65} e_{zx} + 2 S_{66} e_{xy} .
\end{aligned}
\tag{228}
$$

S_{ij}'s are called the influence coefficients.

The number of constants further reduces in the orthotropic case. In such a material, the structures all appear identical after a 180° rotation about any one of the three fixed mutually perpendicular axes. Hence there can be no interaction between the various shear components or the shear and normal components. By taking a plane formed by two of these axes as the x, y plane the number of constants for the orthotropic materials reduces to nine. The stress-strain relations are, thus

$$e_{xx} = C_{11} p_{xx} + C_{12} p_{yy} + C_{13} p_{zz} ,$$

$$e_{yy} = c_{21} p_{xx} + c_{22} p_{yy} + c_{23} p_{zz} ,$$

$$e_{zz} = c_{31} p_{xx} + c_{32} p_{yy} + c_{33} p_{zz} ,$$

$$2 e_{yz} = c_{44} p_{yz} , \quad 2 e_{zx} = c_{55} p_{zx} , \quad 2 e_{xy} = c_{66} p_{xy} . \tag{229}$$

As in isotropic elasticity theory, a simplification occurs in plane strain anisotropic problems. The plane strain deformations in this case also is described by having the displacements in a direction perpendicular to the x, y plane (the plane of symmetry) to be equal to zero. Further the displacements u_x, u_y are functions of x, y only. If this be so, then $e_{zz} = e_{yz} = e_{zx} = 0$. The strain-stress relations can be put in the form

$$\begin{aligned} e_{xx} &= c_{11} p_{xx} + c_{12} p_{yy} + c_{13} p_{zz} + c_{16} p_{xy} , \\ e_{yy} &= c_{21} p_{xx} + c_{22} p_{yy} + c_{23} p_{zz} + c_{26} p_{xy} , \\ e_{zz} &= 0 = c_{31} p_{xx} + c_{32} p_{yy} + c_{33} p_{zz} + c_{36} p_{xy} , \\ 2 e_{xy} &= c_{61} p_{xx} + c_{62} p_{yy} + c_{63} p_{zz} + c_{66} p_{xy} . \end{aligned} \tag{230}$$

From the third equations of (230) we determine the value of p_{zz} . Substituting this value of p_{zz} in the remaining equations the strain-stress relation can be put in the form

$$\frac{\partial u_x}{\partial x} = e_{xx} = \alpha_{11} p_{xx} + \alpha_{12} p_{yy} + \alpha_{16} p_{xy} ,$$

$$\frac{\partial u_y}{\partial y} = e_{yy} = \alpha_{12} p_{xx} + \alpha_{22} p_{yy} + \alpha_{26} p_{xy} ,$$

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 2e_{xy} = \alpha_{61} p_{xx} + \alpha_{62} p_{yy} + \alpha_{66} p_{xy} ,$$

(231)

where

$$\alpha_{11} = \frac{C_{11}C_{33} - C_{13}^2}{C_{33}} , \qquad \alpha_{22} = \frac{C_{22}C_{33} - C_{23}^2}{C_{33}} ,$$

$$\alpha_{12} = \frac{C_{12}C_{33} - C_{13}C_{32}}{C_{33}} , \qquad \alpha_{16} = \frac{C_{16}C_{33} - C_{13}C_{36}}{C_{33}}$$

$$\alpha_{26} = \frac{C_{26}C_{33} - C_{23}C_{36}}{C_{33}} , \qquad \alpha_{66} = C_{66} \cdot \tag{232}$$

In the plane strain orthotropic case the stress strain relations can be put in the form

$$e_{xx} = C_{11} p_{xx} + C_{12} p_{yy} + C_{13} p_{zz} , \qquad e_{yy} = C_{21} p_{xx} + C_{22} p_{yy} + C_{23} p_{zz} ,$$

$$2e_{xy} = C_{66} p_{xy} , \qquad e_{zz} = 0 \equiv C_{31} p_{xx} + C_{32} p_{yy} + C_{33} p_{zz} ,$$

$$e_{xz} = e_{yz} = 0. \tag{233}$$

Although apparantly seven constants appear in these relations (233), the condition that $e_{zz} = 0$ enables to put the value of

p_{zz} in terms of p_{xx}, p_{yy}. Substituting the value of p_{zz}, we obtain

$$e_{xx} = \alpha_{11} p_{xx} + \alpha_{12} p_{yy}, \quad e_{yy} = \alpha_{12} p_{xx} + \alpha_{22} p_{yy},$$

$$2 e_{xy} = \alpha_{66} p_{xy}, \tag{234}$$

where $\alpha_{11}, \alpha_{22}, \alpha_{12}, \alpha_{66}$ are given in equations (232). It may be emphasised that in anisotropic plane strain problems the equations of equilibrium reduce to two (21) and the strain compatibility relations to only one (23).

In the cubic case, the material has equal properties in three orthogonal directions. Hence many of the elastic constants become identical and the strain-stress relations in the general case may be written as

$$e_{xx} = c_{11} p_{xx} + c_{12}(p_{yy} + p_{zz}), \quad 2e_{xy} = c_{66} p_{xy},$$

$$e_{yy} = c_{11} p_{yy} + c_{12}(p_{zz} + p_{xx}), \quad 2e_{xz} = c_{66} p_{zx},$$

$$e_{zz} = c_{11} p_{zz} + c_{12}(p_{xx} + p_{yy}), \quad 2e_{yz} = c_{66} p_{yz}.$$

In the plane strain case , these can be put in the form (234) but in this case $\alpha_{11} = \alpha_{22}$

$$\alpha_{11} = \alpha_{22} = \frac{c_{11}^2 - c_{12}^2}{c_{11}}, \quad \alpha_{12} = \frac{c_{12}(c_{11} - c_{12})}{c_{11}}, \quad \alpha_{66} = c_{66}.$$

It is necessary to identify the constants $\alpha_{11}, \alpha_{22}, \alpha_{12}, \alpha_{66}$ with the constants (Poisson's ratio ν and Young's modulus E) in isotropic case to provide a check on the analysis. The

stress-strain relationship in plane strain deformation in isotropic elasticity can be written as

$$e_{xx} = \frac{(1-\nu^2)}{E} p_{xx} - \frac{\nu(1+\nu)}{E} p_{yy}\,, \quad e_{yy} = \frac{(1-\nu^2)}{E} p_{yy} - \frac{\nu(1+\nu)}{E} p_{xx}\,,$$

$$e_{xy} = \frac{2(1+\nu)}{E} p_{xy}\,.$$

(235)

Thus for the isotropic case

$$\alpha_{11} = \alpha_{22} = \frac{(1-\nu^2)}{E}\,, \quad \alpha_{12} = \frac{-\nu(1+\nu)}{E}\,, \quad \alpha_{66} = \frac{2(1+\nu)}{E}\,. \tag{236}$$

The values of the constants for cubic and isotropic case have been given mainly for comparison. Essentially, we shall be concerned with orthotropic medium. Thus the basic equations will be (6), (8), (13) and (234). The solution, of course, must satisfy the given loading or displacement conditions at the boundary.

The compatibility relations are known in terms of strains (13). These can be put in terms of stresses by using Hooke's Law. In plane strain orthotropic case this reduces to only one. Thus we may summarise the basic equations for plane strain problems in orthotropic medium. Neglecting body forces, the stresses must satisfy the following three equations.

$$\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} = 0\,, \quad \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} = 0\,,$$

$$\alpha_{11}\frac{\partial^2 p_{xx}}{\partial y^2} + \alpha_{22}\frac{\partial^2 p_{yy}}{\partial x^2} + \alpha_{12}\left(\frac{\partial^2 p_{xx}}{\partial x^2} + \frac{\partial^2 p_{yy}}{\partial y^2}\right) = \alpha_{66}\frac{\partial^2 p_{xy}}{\partial x\,\partial y} \qquad (237)$$

with appropriate boundary conditions. The method of obtaining solutions of problems connected with orthotropic elastic medium is discussed in the next chapter.

CHAPTER XVII

Plane Strain Orthotropic Elasticity

The basic equations for plane-strain problems in orthotropic elasticity are given by (237). As in isotropic elasticity the solution of these is obtained by introducing the Airy's stress function χ , which is such that

$$P_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad P_{yy} = \frac{\partial^2 \chi}{\partial x^2}, \quad P_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} . \tag{238}$$

These identically satisfy the first two equations in (237). From the third equation in (237), we obtain the relation

$$\alpha_{22} \frac{\partial^4 \chi}{\partial x^4} + (2\alpha_{12} + \alpha_{66}) \frac{\partial^4 \chi}{\partial x^2 \partial y^2} + \alpha_{11} \frac{\partial^4 \chi}{\partial y^4} = 0 , \tag{239}$$

which is fundamental in the theory of plane orthotropic medium and takes place of the biharmonic equation in isotropic elasticity.

The general solution of this equation depends upon the roots of the characteristic equation

$$\alpha_{11} a^4 + (2\alpha_{12} + \alpha_{66}) a^2 + \alpha_{22} = 0 . \tag{240}$$

If a_1, a_2, a_3, a_4 are the roots of the above equation (240), then the general solution of equation (239) is given by

$$\chi(x, y) = \chi_1(x + a_1 y) + \chi_2(x + a_2 y) + \chi_3(x + a_3 y) + \chi_4(x + a_4 y), \tag{241}$$

where χ_i's are the functions of the variables indicated in the brackets following them.

Green and Zerna [13] have shown on energy considerations that the characteristic equation cannot have real roots. Further the roots will have a relation of the type

$$a_1 = \bar{a}_3 = g_1 + ih_1, \quad a_2 = \bar{a}_4 = g_2 + ih_2, \tag{242}$$

whence equation (241) can be put in the form

$$\chi(x, y) = \chi_1(z_1) + \chi_2(z_2) + \overline{\chi_1(z_1)} + \overline{\chi_2(z_2)}, \tag{243}$$

where

$$z_1 = x + a_1 y = (x + g_1 y) + ih_1 y,$$

$$z_2 = x + a_2 y = (x + g_2 y) + ih_2 y. \tag{244}$$

and the bars indicate the conjugates. χ_1 and χ_2 shall be two analytic functions of z_1 and z_2. For isotropic materials

$$g_1 = g_2 = 0, \qquad h_1 = h_2 = 1 \tag{245}$$

As in isotropic elasticity, a direct method of solving the plane orthotropic problem is based upon the complex variable formulation. The solution of the problem is known if we are able to find the functions χ_1 and χ_2 in (243). These determine the Airy's stress function χ. It is, however, more convenient to deal with two other unknown functions $\varphi(z_1)$ and $\psi(z_2)$ related to χ_1, χ_2 by the equations

$$\varphi(z_1) = \frac{d\chi_1}{dz_1}, \qquad \psi(z_2) = \frac{d\chi_2}{dz_2}. \tag{246}$$

It is obvious that

$$\overline{\varphi(z_1)} = \frac{d\overline{\chi_1}}{d\overline{z_1}} \quad , \quad \overline{\psi(z_2)} = \frac{d\overline{\chi_2}}{d\overline{z_2}} \quad , \tag{247}$$

The stresses p_{xx}, p_{yy} and p_{xy} can be put in terms of φ and ψ as follows.

$$p_{xx} = \frac{\partial^2 \chi}{\partial y^2} = 2\,\mathrm{Re}\left\{a_1^2 \varphi'(z_1) + a_2^2 \psi'(z_2)\right\} ,$$

$$p_{yy} = \frac{\partial^2 \chi}{\partial x^2} = 2\,\mathrm{Re}\left\{\varphi'(z_1) + \psi'(z_2)\right\} ,$$

$$p_{xy} = \frac{-\partial^2 \chi}{\partial x \partial y} = -2\,\mathrm{Re}\left\{a_1 \varphi'(z_1) + a_2 \psi'(z_2)\right\} . \tag{248}$$

Using equations (248), the stress-strain relations (234) and the strain-displacement equations (8), we get

$$u_x(x, y) = 2\,\mathrm{Re}\left\{m_1 \varphi(z_1) + m_2 \psi(z_2)\right\} ,$$

$$u_y(x, y) = 2\,\mathrm{Re}\left\{n_1 \varphi(z_1) + n_2 \psi(z_2)\right\} , \tag{249}$$

where

$$m_1 = \alpha_{11} a_1^2 + \alpha_{12} , \quad m_2 = \alpha_{11} a_2^2 + \alpha_{12}$$

$$n_1 = \frac{\alpha_{12} a_1^2 + \alpha_{22}}{a_1} , \quad n_2 = \frac{\alpha_{12} a_2^2 + \alpha_{22}}{a_2} \tag{250}$$

The constants expressing rigid body rotation have been omitted in the right hand side of expressions (249).

It is necessary to formulate the boundary condition in terms of the functions φ and ψ. For the first fundamental

problem, where the boundary conditions consist of external surface tractions only, the conditions can be formulated as follows. If P_{nx} , P_{ny} be the components of surface traction in x, y directions, we know by equations (71) that

$$P_{nx} = p_{xx} \cos(x, n) + p_{xy} \cos(y, n) ,$$

$$P_{ny} = p_{xy} \cos(x, n) + p_{yy} \cos(y, n) .$$

Using the values of p_{xx} , p_{yy} , p_{xy} from equations (240), we see that

$$2\,\mathrm{Re}\left\{a_1 \varphi(t_1) + a_2 \psi(t_2)\right\} = \int^{s} P_{nx}\, ds = f_1 ,$$

$$2\,\mathrm{Re}\left\{\varphi(t_1) + \psi(t_2)\right\} = -\int^{s} P_{ny}\, ds = f_2 , \qquad (251)$$

where t_1 and t_2 are boundary values of z_1 and z_2 . If P_{nx} and P_{ny} are known at the boundary, in principle the two functions φ and ψ can be determined.

For the second fundamental problem, when the boundary conditions are given in terms of the displacements, the governing equations are determined by equations (249) to be

$$2\,\mathrm{Re}\left\{m_1 \varphi(t_1) + m_2 \psi(t_2)\right\} = u_1(s)$$

$$2\,\mathrm{Re}\left\{n_1 \varphi(t_1) + n_2 \psi(t_2)\right\} = u_2(s) \qquad (252)$$

Hence if u_1 and u_2 are known, φ and ψ may be evaluated.

For regions other than those bounded by a circle, it is convenient to use the conformal mapping techniques. Suppose that an infinite region is bounded internally by a closed curve

(this is the case of an infinite medium with a hole). Let the region outside the curve in z -plane be conformally mapped into the region inside the unit circle $|\zeta|=1$ in ζ-plane by a suitable function $z=\omega(\zeta)$. Obviously $\bar{z}=\overline{\omega(\zeta)}$.

In addition to the z -plane, the z_1 and z_2 planes obtained by an affine transformation

$$z_1 = x + a_1 y = \frac{z+\bar{z}}{2} + a_1 \frac{z-\bar{z}}{2i} ,$$

$$z_2 = x + a_2 y = \frac{z+\bar{z}}{2} + a_2 \frac{z-\bar{z}}{2i} , \tag{253}$$

have to be investigated. The values of z and $\bar{z}$ from $z=\omega(\zeta)$ and $\bar{z}=\overline{\omega(\zeta)}$ are substituted in equations (253) to obtain the values of z_1 and z_2. The functions z_1 and z_2 can be put in terms of ζ by equations $z_1=\omega_1(\zeta)$ and $z_2=\omega_2(\zeta)$.

We now introduce two functions $\mathcal{F}(\zeta)$ and $\mathcal{G}(\zeta)$, which are given by

$$\varphi\{\omega_1(\zeta)\} = \mathcal{F}(\zeta), \quad \psi\{\omega_2(\zeta)\} = \mathcal{G}(\zeta).$$

The two boundary conditions (251) and (252) will therefore become

$$2\mathrm{Re}\{a_1\mathcal{F}(\sigma) + a_2\mathcal{G}(\sigma)\} = f_1(\sigma), \quad 2\mathrm{Re}\{\mathcal{F}(\sigma)+\mathcal{G}(\sigma)\} = f_2(\sigma), \tag{254}$$

and

$$2\mathrm{Re}\{m_1\mathcal{F}(\sigma) + m_2\mathcal{G}(\sigma) = u_1(\sigma), \quad 2\mathrm{Re}\{n_1\mathcal{F}(\sigma)+n_2\mathcal{G}(\sigma)\} = u_2(\sigma), \tag{255}$$

where $\sigma = e^{i\theta}$ is the boundary value of ζ .

The functions $\mathcal{F}(\zeta)$ and $\mathcal{G}(\zeta)$ from either equations (254)

or (255) can be evaluated by the help of Schwartz formula. This states that if $H(\zeta)$ is a holomorphic function within a circle and if $H_1(\sigma) = \mathrm{Re}\, H(\zeta)$ is known on the boundary of a unit circle Γ, then the function $H(\zeta)$ at any point ζ within the circle is given by

$$H(\zeta) = \frac{1}{2\pi i}\int_{\Gamma} H_1(\sigma)\,\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + iA_0 \tag{256}$$

where A_0 is a real constant. Similarly if $H_2(\sigma) = \mathrm{Im}\; H(\zeta)$ is known on the boundary of the unit circle, then the function $H(\zeta)$ at any point ζ within the circle is given by

$$H(\zeta) = \frac{1}{2\pi i}\int i\; H_2(\sigma)\,\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + B_0$$

where B_0 is a real constant.

As already remarked in chapter VII, it is not necessary to map the infinite region bounded internally by a curve to the region within a unit circle. The analysis is also applicable even when the infinite region bounded internally by a curve is mapped to the infinite region bounded internally by the circle. For completeness sake, it is therefore necessary to give the Schwartz formula for such a transformation. These in this case take the forms

$$H(\zeta) = \frac{1}{2\pi i}\int_{\Gamma} H_1(\sigma)\,\frac{(\zeta+\sigma)}{(\zeta-\sigma)}\;\frac{d\sigma}{\sigma} + ih_0\,,$$

$$H(\zeta) = \frac{1}{2\pi i}\int_{\Gamma} i\; H_2(\sigma)\frac{(\zeta+\sigma)}{(\zeta-\sigma)}\;\frac{d\sigma}{\sigma} + g_0\,,$$

where $H_1(\sigma) = \operatorname{Re} H(\zeta)$ and $H_2(\sigma) = \operatorname{Im} H(\zeta)$ on the boundary.

Both of these formulations of the Schwartz lemma applicable to the interior and exterior regions have been used in the literature. However, in the sequel we shall map the region under consideration to the interior of the unit circle.

Finally, as we shall be concerned with the strain energy, it is necessary to describe it very briefly. It may be said that the expression for the strain energy density

$$W = \frac{1}{2} p_{ij} e_{ij} ,$$

remains true in this case also. But the subsequent simplifications for the strain energy given in chapters IV and V would not hold good as they have been derived for the isotropic body by the use of stress-strain relation. For anisotropic case the expression for strain energy can be obtained in terms of strains or stresses by making use of the stress-strain relations (227) or (228). Further these can be simplified for plane case also but as no new principle is involved, we shall not discuss it further. It may also be stated that the Clapeyron's theorem already discussed in chapter IV, relating strain energy to the work done at the boundary surface, still holds.

CHAPTER XVIII

Orthotropic Elasticity: Circular Inclusion Principal Strain

The basic equations of elasticity theory relating to the plane strain orthotropic medium have been given in the previous chapters. We now consider the problem of a circular inclusion in an infinite matrix. The problem in this and four subsequent chapters will relate to plane strain case. As stated in previous chapters, we shall take a plane of symmetry to be the plane under consideration. The stresses, strains and displacements would be obtained at each point of this plane.

The matrix material has been taken to be orthotropic which as special cases includes cubic or isotropic materials. The inclusion material is also taken to be orthotropic (which may be cubic or isotropic) but the elastic properties of this may be different from those of matrix material. We have already introduced the constants $\alpha_{11}, \alpha_{22}, \alpha_{12}$ and α_{66} in equations (234) which specify the elastic properties of the material. Here to distinguish the elastic properties of inclusion from those of the matrix, we use $\beta_{11}, \beta_{22}, \beta_{12}, \beta_{66}$ for the inclusion material.

Consider a circular region of radius a in an infinite unstressed continuous orthotropic medium. Let it tend to undergo a non-elastic displacement to an elliptic shape of semi-axes $a(1+\delta_a)$ and $a(1+\delta_b)$ in the absence of the matrix. The possible equilibrium boundary within the matrix may be taken as an ellipse of semi-axes $a(1+\epsilon_1)$ and $a(1+\epsilon_2)$.

It may be remarked here that great care should be taken

for the assumption of equilibrium boundary. For isotropic case, it is true that if the inclusion undergoes a non-elastic displacement to a concentric cylinder of radius $a(1+\delta)$ the equilibrium boundary will also be a cylinder of radius $a(1+\epsilon)$. However, this is not true in orthotropic case. As we shall see subsequently the equilibrium boundary will be an ellipse. Hence we consider the more general problem when the free surface of the inclusion has been taken to be an ellipse.

The internal non-elastic displacements in the inclusion corresponding to the non-elastic boundary displacements are given by $\delta_a x$, $\delta_b y$. The final displacements measured from its initial size are $\epsilon_1 x$, $\epsilon_2 y$. Hence the elastic displacements in the inclusion can be taken as

$$U_x = (\epsilon_1 - \delta_a) x \ , \quad U_y = (\epsilon_2 - \delta_b) y$$

Therefore, the strains are

$$\mathcal{E}_{xx} = (\epsilon_1 - \delta_a) \ , \quad \mathcal{E}_{yy} = (\epsilon_2 - \delta_b) \ , \quad \mathcal{E}_{xy} = 0 \ . \tag{257}$$

By the stress-strain relations (233) the corresponding stresses are

$$P_{xx} = \frac{(\epsilon_1 - \delta_a)\beta_{22} - (\epsilon_2 - \delta_b)\beta_{12}}{\beta_{11}\beta_{22} - \beta_{12}^2} \ , \quad P_{yy} = \frac{(\epsilon_2 - \delta_b)\beta_{11} - (\epsilon_1 - \delta_a)\beta_{12}}{\beta_{11}\beta_{22} - \beta_{12}^2} \ , \quad P_{xy} = 0 \ . \tag{258}$$

By putting the values of the strain and stress components from equations (257) and (258) respectively in equation (40), we obtain the strain energy density in the inclusion to be

$$\frac{1}{2} P_{ij} \mathcal{E}_{ij} = \frac{(\epsilon_1 - \delta_a)^2 \beta_{22} + (\epsilon_2 - \delta_b)^2 \beta_{11} - 2(\epsilon_1 - \delta_a)(\epsilon_2 - \delta_b)\beta_{12}}{2(\beta_{11}\beta_{22} - \beta_{12}^2)} .$$

Hence the strain energy in the inclusion per unit thickness is

$$W_I = \frac{\pi a^2 \{(\epsilon_1 - \delta_a)^2 \beta_{22} + (\epsilon_2 - \delta_b)^2 \beta_{11} - 2(\epsilon_1 - \delta_a)(\epsilon_2 - \delta_b)\beta_{12}\}}{2(\beta_{11}\beta_{22} - \beta_{12}^2)} . \tag{259}$$

We now consider the case of the matrix. The displacement of the interior boundary of the matrix is given by

$$u_x = \epsilon_1 x = \epsilon_1 \frac{z + \bar{z}}{2} , \quad u_y = \epsilon_2 y = \epsilon_2 \frac{z - \bar{z}}{2i} . \tag{260}$$

The function

$$z = \frac{a}{\zeta} , \tag{261}$$

maps the region outside and on the circle of radius a in the z-plane to a region within and on a unit circle in the ζ-plane.

The boundary conditions (260) are given in terms of displacements. Substituting these in equations (255)

$$\begin{aligned} 2\,\mathrm{Re}\{m_1 \mathcal{F}(\sigma) + m_2 \mathcal{G}(\sigma)\} &= u_1(\sigma) = \frac{a\epsilon_1}{2}\left(\sigma + \frac{1}{\sigma}\right) , \\ 2\,\mathrm{Re}\{n_1 \mathcal{F}(\sigma) + n_2 \mathcal{G}(\sigma)\} &= u_2(\sigma) = \frac{ia\epsilon_2}{2}\left(\sigma - \frac{1}{\sigma}\right) . \end{aligned} \tag{262}$$

Multiplying both sides of these two equations by $(\sigma + \zeta)\,d\sigma / 2\pi i(\sigma - \zeta)\sigma$, and making use of Schwartz formula (256), we

get (neglecting the constants of integration, which do not affect the stresses)

$$m_1 \mathcal{F}(\zeta) + m_2 \mathcal{G}(\zeta) = \frac{a\epsilon_1}{2}\,\zeta$$

$$n_1 \mathcal{F}(\zeta) + n_2 \mathcal{G}(\zeta) = \frac{i a \epsilon_2 \zeta}{2}\,.$$

whence

$$\mathcal{F}(\zeta) = \frac{n_2\epsilon_1 - i m_2 \epsilon_2}{2(m_1 n_2 - m_2 n_1)}\, a\zeta, \quad \mathcal{G}(\zeta) = \frac{-(n_1\epsilon_1 - i m_1\epsilon_2)}{2(m_1 n_2 - m_2 n_1)}\,. \tag{263}$$

It is now possible to evaluate the functions $\varphi(z_1)$ and $\psi(z_2)$, if the transforming functions $z_1 = \omega_1(\zeta), z_2 = \omega_2(\zeta)$ are known. For this it is necessary to find $\omega_1(\zeta)$ and $\omega_2(\zeta)$ which are found as follows. On the boundary where $\zeta = \sigma$,

$$z_1 = x + a_1 y = \frac{a(1+ia_1)}{2}\sigma + \frac{a(1-ia_1)}{2}\frac{1}{\sigma}\,,$$

$$z_2 = x + a_2 y = \frac{a(1+ia_2)}{2}\sigma + \frac{a(1-ia_2)}{2}\frac{1}{\sigma}\,,$$

and hence the transforming functions $\omega_1(\zeta)$ and $\omega_2(\zeta)$ are

$$z_1 = \omega_1(\zeta) = \frac{a(1+ia_1)}{2}\zeta + \frac{a(1-ia_1)}{2}\frac{1}{\zeta}\,,$$

$$z_2 = \omega_2(\zeta) = \frac{a(1+ia_2)}{2}\zeta + \frac{a(1-ia_2)}{2}\frac{1}{\zeta}\,. \tag{264}$$

The stresses can now be determined by equations (248) and the displacements by (249). However, these expressions will be

very complicated if the choice of the coordinate axes is arbitrary. This may be simplified by a proper choice of the axes.

Choosing the coordinate axes of the system to be given by lines of intersection of the planes of elastic symmetry of the material, the roots of the characteristic equation (240) can be put as

$$a_1 = i h_1 , \qquad a_2 = i h_2$$

The constants m_1, m_2, n_1, n_2 will therefore be

$$m_1 = -\alpha_{11} h_1^2 + \alpha_{12} , \qquad m_2 = -\alpha_{11} h_2^2 + \alpha_{12} ,$$

$$n_1 = \frac{-\alpha_{12} h_1^2 + \alpha_{22}}{i h_1} \qquad n_2 = \frac{-\alpha_{12} h_2^2 + \alpha_{22}}{i h_2} \tag{265}$$

Also from equation (240) $\alpha_{11} h_1^2 h_2^2 = \alpha_{22}$.

Substituting the values of the constants m_1, m_2, n_1, n_2 in (263), we can write the two complex functions $\mathcal{F}(\varsigma)$ and $\mathcal{G}(\varsigma)$ as

$$\mathcal{F}(\varsigma) = \frac{h_1\{(\alpha_{22} - \alpha_{12} h_2^2)\epsilon_1 + h_2(\alpha_{12} - \alpha_{11} h_2^2)\epsilon_2\} \, a\,\varsigma}{2(h_1 - h_2)\{2\alpha_{12}\alpha_{22} + \alpha_{12}^2 h_1 h_2 - \alpha_{11}\alpha_{22}(h_1^2 + h_1 h_2 + h_2^2)\}} ,$$

$$= \frac{c_1 a \varsigma}{2(h_1 - h_2) d} ,$$

$$\mathcal{G}(\varsigma) = -\frac{h_2\{(\alpha_{22} - \alpha_{12} h_1^2)\epsilon_1 + h_1(\alpha_{12} - \alpha_{11} h_1^2)\epsilon_2\} \, a\,\varsigma}{2(h_1 - h_2)\{2\alpha_{12}\alpha_{22} + \alpha_{12}^2 h_1 h_2 - \alpha_{11}\alpha_{22}(h_1^2 + h_1 h_2 + h_2^2)} ,$$

$$= -\frac{c_2 a \varsigma}{2(h_1 - h_2) d} ,$$

(266)

where

$$c_1 = h_1\left\{(\alpha_{22}-\alpha_{12}h_2^2)\,\epsilon_1 + h_2(\alpha_{12}-\alpha_{11}h_2^2)\,\epsilon_2\right\},$$

$$c_2 = h_2\left\{(\alpha_{22}-\alpha_{12}h_1^2)\,\epsilon_1 + h_1(\alpha_{12}-\alpha_{11}h_1^2)\,\epsilon_2\right\},$$

$$d = 2\alpha_{12}\alpha_{22} + \alpha_{12}^2 h_1 h_2 - \alpha_{11}\alpha_{22}(h_1^2 + h_1 h_2 + h_2^2\} . \tag{267}$$

The stresses in the matrix can now be determined by equations (248) which involve $\varphi(z_1)$ and $\psi(z_2)$. These can be evaluated by equations (266) and (264) and are given by

$$\varphi'(z_1) = \frac{dF}{d\zeta}\cdot\frac{d\zeta}{dz_1} = \frac{c_1}{(h_1-h_2)d}\quad\frac{\zeta^2}{\{(1-h_1)\zeta^2-(1+h_1)\}},$$

$$\psi'(z_2) = \frac{dG}{d\zeta}\cdot\frac{d\zeta}{dz_2} = \frac{-c_2}{(h_1-h_2)d}\quad\frac{\zeta^2}{\{(1-h_2)\zeta^2-(1+h_2)\}} . \tag{268}$$

Thus the values of the stresses and corresponding strains can be evaluated. Having known the stresses and strainsit is possible to find the strain energy in the matrix in principle.

It is, however, less complicated to make use of the Clapeyron's theorem. This involves the determination of the values of the stresses at the boundary. By substituting $\zeta = e^{i\theta}$ in equations (268) and then making use of equations (248), we get

$$\overset{c}{p}_{xx} = \frac{-1}{(h_1-h_2)d}\left[\frac{c_1h_1^2\{(1-h_1)-(1+h_1)\cos 2\theta\}}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{c_2h_2^2\{(1-h_2)-(1+h_2)\cos 2\theta\}}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right]$$

$$\overset{c}{p}_{yy} = \frac{1}{(h_1-h_2)d}\left[\frac{c_1\{(1-h_1)-(1+h_1)\cos 2\theta\}}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{c_2\{(1-h_2)-(1+h_2)\cos 2\theta\}}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right]$$

$$\overset{c}{p}_{xy} = \frac{-1}{(h_1 - h_2)d}\left[\frac{c_1 h_1(1+h_1)\sin 2\theta}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{c_2 h_2(1+h_2)\sin 2\theta}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right].$$

(269)

Making use of the formulae (269), (71) together with the values of the displacements given by (260), it can be shown that the strain energy W_m in the matrix is given by

$$W_m = \frac{1}{2}\int (P_{nx} u_x + P_{ny} u_y)\, ds$$

$$= \frac{a^2}{4}\int_0^{2\pi}\{\overset{c}{p}_{xy}(\epsilon_1+\epsilon_2)\sin 2\theta - \overset{c}{p}_{xx}\epsilon_1(1+\cos 2\theta) - \overset{c}{p}_{yy}\epsilon_2(1-\cos 2\theta)\}\, d\theta$$

$$= \frac{-\pi a^2}{2d}\{\epsilon_1^2\alpha_{22}(h_1+h_2) + 2(\alpha_{22}+\alpha_{12}h_1h_2)\epsilon_1\epsilon_2 + \epsilon_2^2\alpha_{11}h_1h_2(h_1+h_2)\}.$$

(270)

The calculation proceeds as in the case of isotropic matrix. Some of the relevant integrals are given in appendix III.

Adding the strain energy of the inclusion (259) to that of the matrix (270), the total strain energy is

$$W = W_I + W_m = \frac{\pi a^2}{2}\left[\frac{(\epsilon_1-\delta_a)^2\beta_{22} + (\epsilon_2-\delta_b)^2\beta_{11} - 2(\epsilon_1-\delta_a)(\epsilon_2-\delta_b)\beta_{12}}{\beta_{11}\beta_{22}-\beta_{12}^2}\right]$$

$$+ \frac{\pi a^2}{-2d}\left[\epsilon_1^2\alpha_{22}(h_1+h_2) + 2(\alpha_{22}+\alpha_{12}h_1h_2)\epsilon_1\epsilon_2 + \epsilon_2^2\alpha_{11}h_1h_2(h_1+h_2)\right].$$

(271)

Minimising W with respect to ϵ_1 and ϵ_2 i.e., setting $\partial W/\partial\epsilon_1 = 0$, $\partial W/\partial\epsilon_2 = 0$ we obtain

$$\epsilon_1 = \frac{\left[\delta_a\{\beta_{22}\alpha_{11}h_1h_2(h_1+h_2) - d + \beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)\} - \delta_b\{\alpha_{11}\beta_{12}h_1h_2(h_1+h_2) + \beta_{11}(\alpha_{22}+\alpha_{12}h_1h_2)\}\right]}{\left[h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2) - d + \alpha_{22}\beta_{11}(h_1+h_2) + \alpha_{11}\beta_{22}h_1h_2(h_1+h_2) + 2\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)\right]},$$

$$\epsilon_2 = \frac{\left[\delta_b\{\beta_{11}\alpha_{22}(h_1+h_2) - d + \beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)\} - \delta_a\{\beta_{12}\alpha_{22}(h_1+h_2) + \beta_{22}(\alpha_{22}+\alpha_{12}h_1h_2)\}\right]}{h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2) - d + \alpha_{22}\beta_{11}(h_1+h_2) + \alpha_{11}\beta_{22}h_1h_2(h_1+h_2) + 2\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)}$$

(272)

The determination of the equilibrium boundary is now a matter of substitution of equations (272) in (260). The elastic field everywhere may be determined by substituting ϵ_1 ,ϵ_2 in the relevant equations.

Using the formulae (117) and setting the values of P_{xx}, P_{yy}, P_{xy} of inclusion in it and the values of p^c_{xx}, p^c_{yy}, p^c_{xy} at the boundary of the matrix, it can be shown that

$$P^b_{mn} = p^b_{mn} = \frac{\{C_1(1+h_1) - C_2(1+h_2)\} - \{C_1(1-h_1) - C_2(1-h_2)\}\cos 2\theta}{2(h_1-h_2)d},$$

$$P^b_{ns} = p^b_{ns} = \frac{\{C_1(1-h_1) - C_2(1-h_2)\}\sin 2\theta}{2(h_1-h_2)d},$$

$$P^b_{ss} = \frac{\{C_1(1+h_1) - C_2(1+h_2)\} + \{C_1(1-h_1) - C_2(1-h_2)\}\cos 2\theta}{2(h_1-h_2)d},$$

$$p^b_{ss} = P^b_{ss} - \frac{2}{(h_1-h_2)d}\left[\frac{C_1h_1(1+h_1)}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{C_2h_2(1+h_2)}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right],$$

(273)

where the constants C_1 and C_2 are evaluated by substituting the values of ϵ_1, ϵ_2 from equations (272) in (267). These are given by

$$C_1 = \frac{-h_1 d\left[\delta_a\{\beta_{22}+\beta_{12}h_2-\alpha_{12}h_2^2+\alpha_{22}\}+\delta_b h_2\{\alpha_{12}-\alpha_{11}h_1^2-(\beta_{11}h_2+\beta_{12})\}\right]}{\left[h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2)-d+\alpha_{22}\beta_{11}(h_1+h_2)+\alpha_{11}\beta_{22}h_1h_2(h_1+h_2)+2\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)\right]},$$

$$C_2 = \frac{-h_2 d\left[\delta_a\{\beta_{22}+\beta_{12}h_1-\alpha_{12}h_1^2+\alpha_{22}\}+\delta_b h_1\{\alpha_{12}-\alpha_{11}h_2^2-(\beta_{11}h_1+\beta_{12})\}\right]}{h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2)-d+\alpha_{22}\beta_{11}(h_1+h_2)+\alpha_{11}\beta_{22}h_1h_2(h_1+h_2)+2\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)}.$$

(274)

The jump in the hoop stress is given by

$$p_{ss}^{b}-p_{ss}^{b} = -\frac{2}{(h_1-h_2)d}\left[\frac{C_1h_1(1+h_1)}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}}-\frac{C_2h_2(1+h_2)}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right].$$

From equations (273) it can be seen that the normal and shear stresses are continuous at the interface. In the analysis we had already assumed the continuity of the displacements.

Similarly for the case of matrix, the boundary displacements are

$$u_x = \gamma_1 y = \gamma_1 \frac{z-\bar{z}}{2i}, \qquad u_y = \gamma_2 x = \gamma_2 \frac{z+\bar{z}}{2}$$

The mapping function $z = a/\zeta$ given by (261) is used to map the infinite region outside the hole in the z-plane to that within the unit circle in the ζ-plane. The boundary conditions given in terms of displacements are used in equations (255) to obtain the following two equations for the evaluation of $\mathcal{F}(\zeta)$ and $\mathcal{G}(\zeta)$.

$$2\mathrm{Re}\left\{m_1 \mathcal{F}(\sigma) + m_2 \mathcal{G}(\sigma)\right\} = \frac{ia\gamma_1}{2}\left(\sigma - \frac{1}{\sigma}\right),$$

$$2\mathrm{Re}\left\{n_1 \mathcal{F}(\sigma) + n_2 \mathcal{G}(\sigma)\right\} = \frac{a\gamma_2}{2}\left(\sigma - \frac{1}{\sigma}\right).$$

By the methods analogous to the one used in previous chapter it can be readily shown that

$$m_1 \mathcal{F}(\zeta) + m_2 \mathcal{G}(\zeta) = \frac{ia\gamma_1}{2}\zeta,$$

$$n_1 \mathcal{F}(\zeta) + n_2 \mathcal{G}(\zeta) = \frac{a\gamma_2}{2}\zeta.$$

Solving for $\mathcal{F}(\zeta)$ and $\mathcal{G}(\zeta)$, we get

$$\mathcal{F}(\zeta) = \frac{(in_2\gamma_1 - m_2\gamma_2)}{2(m_1 n_2 - m_2 n_1)} a\zeta, \quad \mathcal{G}(\zeta) = \frac{-(in_1\gamma_1 - m_1\gamma_2)}{2(m_1 n_2 - m_2 n_1)} a\zeta$$

These complex functions together with the transforming functions $\omega_1(\varsigma)$ and $\omega_2(\varsigma)$ given by (264) determine the elastic field completely in the matrix. However, as already remarked, it is more complicated to determine it for an arbitrary choice of coordinate axes. Here again, by the same choice of axes as in the case of principal strains, the complex functions may be simplified as

$$\mathcal{F}(\varsigma) = \frac{i\,c_1'\,a\,\varsigma}{2(h_1-h_2)\,d}\,, \qquad \mathcal{G}(\varsigma) = \frac{-i\,c_2'\,a\,\varsigma}{2(h_1-h_2)\,d}\,, \tag{279}$$

where

$$\begin{aligned}
c_1' &= -h_1\left\{(\alpha_{22}-\alpha_{12}h_2^2)\,\gamma_1 - h_2(\alpha_{12}-\alpha_{11}h_2^2)\,\gamma_2\right\}\,,\\
c_2' &= -h_2\left\{(\alpha_{22}-\alpha_{12}h_1^2)\,\gamma_1 - h_1(\alpha_{12}-\alpha_{11}h_1^2)\,\gamma_2\right\}\,,\\
d &= 2\alpha_{12}\alpha_{22} + \alpha_{12}^2 h_1h_2 - \alpha_{11}\alpha_{22}(h_1^2+h_1h_2+h_2^2)\,.
\end{aligned} \tag{280}$$

Making use of the mapping functions and transforming the functions $\mathcal{F}(\varsigma)$ and $\mathcal{G}(\varsigma)$ to $\varphi(z_1)$ and $\psi(z_2)$ and then using the relations between the stresses and complex potential functions as given by (248), it can be shown that at the boundary

$$\begin{aligned}
p_{xx}^c &= -\frac{\sin 2\theta}{(h_1-h_2)\,d}\left[\frac{c_1'h_1^2(1+h_1)}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{c_2'h_2^2(1+h_2)}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right],\\
p_{yy}^c &= \frac{\sin 2\theta}{(h_1-h_2)\,d}\left[\frac{c_1'(1+h_1)}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{c_2'(1+h_2)}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right],
\end{aligned}$$

$$p^c_{xy} = \frac{1}{(h_1-h_2)d}\left[\frac{c'_1 h_1\{(1-h_1)-(1+h_1)\cos 2\theta\}}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{c'_2 h_2\{(1-h_2)-(1+h_2)\cos 2\theta\}}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right]. \tag{281}$$

The strain energy in the matrix is calculated by the methods used in previous chapters. In this case we shall get

$$W_m = \frac{a^2}{4}\int\left[p^c_{xx}\gamma_1 \sin 2\theta + p^c_{yy}\gamma_2 \sin 2\theta - 2p^c_{xy}(\gamma_1 \sin^2\theta + \gamma_2 \cos^2\theta\right]d\theta \tag{282}$$

where the integration is to be carried out on the boundary of the circle in z-plane. Setting the values of p^c_{xx}, p^c_{yy}, p^c_{xy} and from equations (281) in this and integrating, we get

$$W_m = \frac{\pi a^2}{-2d}\left[\alpha_{22}(h_1+h_2)\gamma_1^2 - 2(\alpha_{22}+\alpha_{12}h_1h_2)\gamma_1\gamma_2 + \alpha_{11}h_1h_2(h_1+h_2)\gamma_2^2\right]. \tag{283}$$

Some of the relevant integrals needed for this are given in appendix III.

The total strain energy in the combined system is thus given by

$$W = W_I + W_m = \frac{\pi a^2}{2}\left[\frac{(\gamma_1+\gamma_2-\gamma_a-\gamma_b)^2}{\beta_{66}} + \frac{1}{-d}\left\{\alpha_{22}(h_1+h_2)\gamma_1^2 - 2(\alpha_{22}+\alpha_{12}h_1h_2)\gamma_1\gamma_2 + \alpha_{11}h_1h_2(h_1+h_2)\gamma_2^2\right\}\right]. \tag{284}$$

Minimising W with respect to γ_1 and γ_2 we get

$$\gamma_1 = \frac{(\gamma_a+\gamma_b)\{\alpha_{11}(h_1+h_1h_2+h_2)+\alpha_{12}\}}{\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{11}(h_1+h_1h_2+h_2)+2\alpha_{12}\}+\beta_{66}},$$

$$\gamma_2 = \frac{(\gamma_a+\gamma_b)\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{12}\}}{\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{11}(h_1+h_1h_2+h_2)+2\alpha_{12}\}+\beta_{66}}. \tag{285}$$

From these values of γ_1, γ_2 the solution is completely known. These satisfy the stress continuity requirements also. At the interface, we shall have for the normal, shear and hoop stresses

$$P^{b}_{nn} = p^{b}_{nn} = \frac{-(\gamma_a+\gamma_b)\sin 2\theta}{\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{11}(h_1+h_1h_2+h_2)+2\alpha_{12}\}+\beta_{66}},$$

$$P^{b}_{ns} = p^{b}_{ns} = \frac{-(\gamma_a+\gamma_b)\cos 2\theta}{\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{11}(h_1+h_1h_2+h_2)+2\alpha_{12}\}+\beta_{66}},$$

$$P^{b}_{ss} = -P^{b}_{nn} = \frac{(\gamma_a+\gamma_b)\sin 2\theta}{\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{11}(h_1+h_1h_2+h_2)+2\alpha_{12}\}+\beta_{66}},$$

$$p^{b}_{ss} = P^{b}_{ss} + \left[\frac{c_1'(1-h_1^2)(1+h_1)}{\{(1+h_1^2)-(1-h_1^2)\cos 2\theta\}} - \frac{c_2'(1-h_2^2)(1+h_2)}{\{(1+h_2^2)-(1-h_2^2)\cos 2\theta\}}\right], \tag{286}$$

where

$$c_1' = \frac{-h_1(\gamma_a+\gamma_b)\left[(\alpha_{22}-\alpha_{12}h_2^2)\{\alpha_{11}(h_1+h_1h_2+h_2)+\alpha_{12}\} - (\alpha_{12}-\alpha_{11}h_2^2)\,h_2\{\alpha_{11}(h_1+h_2+1)h_1h_2+\alpha_{12}\}\right]}{\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{11}(h_1+h_1h_2+h_2)+2\alpha_{12}\}+\beta_{66}},$$

$$c_2' = \frac{-h_2(\gamma_a+\gamma_b)\left[(\alpha_{22}-\alpha_{12}h_1^2)\{\alpha_{11}(h_1+h_1h_2+h_2)+\alpha_{12}\} - h_1(\alpha_{12}-\alpha_{11}h_1^2)\{\alpha_{11}(h_1+h_2+1)h_1h_2+\alpha_{12}\}\right]}{\{\alpha_{11}h_1h_2(h_1+h_2+1)+\alpha_{11}(h_1+h_1h_2+h_2)+2\alpha_{12}\}+\beta_{66}}. \tag{287}$$

It can be seen from equations (286) that the values of the normal and hoop stresses are zero at $\theta = 0, \pi/2, \pi$ and $3\pi/2$ where the shear stresses are maximum and equal to one another in magnitude. The sign of the hoop stresses in the inclusion in the first and third quadrants of the circle will be opposite to that in the second and fourth quadrants. The sense of this hoop stress in the matrix will be of opposite sense at every point (except at those points where hoop stress is zero).

The values of γ_1 , γ_2 have been evaluated through the use of the equations of equilibrium, the compatibility conditions and the stress-strain relations. They satisfy the continuity requirements of displacements at the interface. It should be remembered that for determining the continuity of displacements it is the net displacement of inclusion that should be considered.

The strain energy in the inclusion may now be evaluated by equations (278) and (285). Similarly the strain energy in the matrix may be obtained by setting the values of γ_1, γ_2 from equations (285) in (283).

It is interesting to notice that β_{66} is the only constant present in the equations (285) and (286) and that the constants $\beta_{11}, \beta_{22}, \beta_{12}$ do not play any part whatsoever. Thus the elastic constant β_{66} can be found experimentally if we know γ_1 or γ_2 and $(\gamma_a+\gamma_b)$ by the relations (285). Of course, we should know the elastic constants of the matrix material. The condition of a non-elastic displacement components $\gamma_a y, \gamma_b x$ parallel to co-ordinate axes directions can be simulated by suitable

experimental conditions and γ_1, γ_2 may be measured.

An interesting result to note is that when the inclusion is rigid (i.e., $\beta_{66} = 0$) $\gamma_1 + \gamma_2 = \gamma_a + \gamma_b$ which indicates that the shear strain is unaltered. However, the initial rigid body rotation $\gamma_a - \gamma_b$ is changed to

$$\gamma_1 - \gamma_2 = \frac{(\gamma_a + \gamma_b)\,\alpha_{11}(h_1 + h_2)\,(1 - h_1 h_2)}{\{\alpha_{11} h_1 h_2 (h_1 + h_2 + 1) + \alpha_{11}(h_1 + h_1 h_2 + h_2) + 2\alpha_{12}\}}$$

If the inclusion is undergoing a spontaneous non-elastic displacement $(\delta_a x + \gamma_a y), (\delta_b y + \gamma_b x)$ in the x and y directions, then the stresses, strains and displacements are obtained by the super-position of the results in this and the previous chapter. The equilibrium boundary in such a case will be obtained by a displacement

$$u_x = \epsilon_1 x + \gamma_1 y\,, \quad u_y = \epsilon_2 y + \gamma_2 x$$

of the boundary of the hole, where ϵ_1, ϵ_2 are given by equations (272) and γ_1, γ_2 by (285).

CHAPTER XX

Elliptic Inclusion Problem in Orthotropic Elasticity Principal Strain

Let an orthotropic elliptic region with major and minor axes $2a$ and $2b$ tend to attain a free surface of an ellipse of axes $2a(1+\delta_a)$ and $2b(1+\delta_b)$ without any rotation of axes or slipping. The values of δ_a, δ_b are taken to be such that they are within the proportional limit. We assume that the equilibrium interface is an ellipse of semi-axes $a(1+\epsilon_1)$ and $b(1+\epsilon_2)$ (the axes coinciding with that of the hole). ϵ_1, ϵ_2 are the unknowns to be determined. We shall show that the values of ϵ_1, ϵ_2 obtained from the subsequent analysis, satisfy all the boundary conditions as well as the equations of elasticity.

The elastic displacement field in the inclusion is calculated from the free surface to its equilibrium position. It is given by

$$U_x = (\epsilon_1 - \delta_a)\, x, \quad U_y = (\epsilon_2 - \delta_b)\, y .$$

Therefore the strains are

$$\mathcal{E}_{xx} = (\epsilon_1 - \delta_a), \quad \mathcal{E}_{yy} = (\epsilon_2 - \delta_b), \quad \mathcal{E}_{xy} = 0, \tag{288}$$

and the corresponding stresses are given by

$$P_{xx} = \frac{(\epsilon_1 - \delta_a)\beta_{22} - (\epsilon_2 - \delta_b)\beta_{12}}{\beta_{11}\beta_{22} - \beta_{12}^2}, \quad P_{yy} = \frac{(\epsilon_2 - \delta_b)\beta_{11} - (\epsilon_1 - \delta_a)\beta_{12}}{\beta_{11}\beta_{22} - \beta_{12}^2}, \quad P_{xy} = 0 \tag{289}$$

where we use β_{11}, β_{22}, β_{12}, β_{66} for the elastic constants of the inclusion material as in the previous chapter. The strain

energy density is obtained from equations (288) and (289) and is

$$\frac{1}{2}\left[\frac{(\epsilon_1-\delta_a)^2\beta_{22}+(\epsilon_2-\delta_b)^2\beta_{11}-2(\epsilon_1-\delta_a)(\epsilon_2-\delta_b)\beta_{12}}{\beta_{11}\beta_{22}-\beta_{12}^2}\right],$$

whence the strain energy in the inclusion is given by

$$W_I=\frac{\pi ab}{2}\left[\frac{(\epsilon_1-\delta_a)^2\beta_{22}+(\epsilon_2-\delta_b)^2\beta_{11}-2(\epsilon_1-\delta_a)(\epsilon_2-\delta_b)\beta_{12}}{\beta_{11}\beta_{22}-\beta_{12}^2}\right]. \qquad (290)$$

The displacement components of the interior boundary of the matrix are given by

$$u_x=\epsilon_1 x=\epsilon_1\frac{z+\bar{z}}{2}, \quad u_y=\epsilon_2 y=\frac{z-\bar{z}}{2i}\epsilon_2 . \qquad (291)$$

The function mapping the region outside the ellipse in the z-plane to a region within a unit circle $|\zeta|=1$ in the ζ plane is the same which we have used in equation (190). We have put the values of R and m in this equation in terms of a and b. Thus the mapping function is

$$z=\omega(\zeta)=\frac{(a+b)}{2\zeta}+\frac{(a-b)}{2}\zeta . \qquad (292)$$

Therefore the transforming functions from z_1 and z_2 planes are

$$z_1=\frac{a+ia_1b}{2}\zeta+\frac{a-ia_1b}{2}\frac{1}{\zeta},$$

$$z_2=\frac{a+ia_2b}{2}\zeta+\frac{a-ia_2b}{2}\frac{1}{\zeta}, \qquad (293)$$

where the same type of arguments are used as have been used in chapter XVIII.

The boundary displacement components u_x, u_y are given by (291). Further z, $\bar{z}$ in these equations can be replaced in terms of ζ, $\bar{\zeta}$ by equation (292). At the boundary $\zeta = \sigma$ and $\sigma\bar{\sigma} = 1$. Hence transforming z in terms of σ and using equations (255),

$$2\,\mathrm{Re}\left\{m_1 \mathcal{F}(\sigma) + m_2 \mathcal{G}(\sigma)\right\} = \frac{\epsilon_1 a}{2}\left(\sigma + \frac{1}{\sigma}\right),$$

$$2\,\mathrm{Re}\left\{n_1 \mathcal{F}(\sigma) + n_2 \mathcal{G}(\sigma)\right\} = \frac{i\epsilon_2 b}{2}\left(\sigma - \frac{1}{\sigma}\right). \tag{294}$$

Multiplying both sides of equations (294) by $(\sigma + \zeta)\,d\sigma / 2\pi i(\sigma - \zeta)\sigma$ and integrating round the contour of the unit circle, and making use of the Schwartz formula, we evaluate the complex potential functions $\mathcal{F}(\zeta)$ and $\mathcal{G}(\zeta)$. These are

$$\mathcal{F}(\zeta) = \frac{a n_2 \epsilon_1 - i b m_2 \epsilon_2}{2(m_1 n_2 - m_2 n_1)}\zeta, \quad \mathcal{G}(\zeta) = \frac{-(a n_1 \epsilon_1 - i b m_1 \epsilon_2)}{2(m_1 n_2 - m_2 n_1)}\zeta \tag{295}$$

As already stated the coordinate axes are the lines of intersection of planes of elastic symmetry of the material. Then the roots of the characteristic equation (240) may be written as $a_1 = ih_1$, $a_2 = ih_2$ and the values of m_1, m_2, n_1, n_2 are given by equations(265). Further $\alpha_{11} h_1^2 h_2^2 = \alpha_{22}$. Making use of these relations the values of $\mathcal{F}(\zeta)$ and $\mathcal{G}(\zeta)$ in equations (295) may be simplified to

$$\mathcal{F}(\zeta) = \frac{c_1 \zeta}{2(h_1 - h_2)d}, \quad \mathcal{G}(\zeta) = \frac{-c_2 \zeta}{2(h_1 - h_2)d}, \tag{296}$$

where

$$c_1 = a\epsilon_1(\alpha_{22} - \alpha_{12}h_2^2)\,h_1 + b\epsilon_2 h_1 h_2(\alpha_{12} - \alpha_{11}h_2^2),$$

$$c_2 = a\epsilon_1(\alpha_{22} - \alpha_{12}h_1^2)\,h_2 + b\epsilon_2 h_1 h_2(\alpha_{12} - \alpha_{11}h_1^2),$$

$$d = 2\alpha_{12}\alpha_{22} + \alpha_{12}^2 h_1 h_2 - \alpha_{11}\alpha_{22}(h_1 + h_1 h_2 + h_2). \qquad (297)$$

It is now possible to determine the stress-strain field everywhere in the matrix by following the procedure already stated in chapter XVIII.

The next step is to calculate the strain energy in the matrix. As already remarked, it is difficult to obtain the energy in the matrix by determining the stresses and strains everywhere. We therefore again use Clapeyron's theorem. For it we need to determine the boundary tractions, and therefore the boundary stresses. Denoting these by $\overset{c}{p}_{xx}$, $\overset{c}{p}_{yy}$ and $\overset{c}{p}_{xy}$ and making use of equations (248), (293) and (296), we get

$$\overset{c}{p}_{xx} = \frac{-1}{(h_1-h_2)d}\left[\frac{c_1h_1^2\{(a-h_1b)-(a+h_1b)\cos 2\theta\}}{\{(a^2+h_1^2b^2)-(a^2-h_1^2b^2)\cos 2\theta\}} - \frac{c_2h_2^2\{(a-h_2b)-(a+h_2b)\cos 2\theta\}}{\{(a^2+h_2^2b^2)-(a^2-h_2^2b^2)\cos 2\theta\}}\right],$$

$$\overset{c}{p}_{yy} = \frac{1}{(h_1-h_2)d}\left[\frac{\{(a-h_1b)-(a+h_2b)\cos 2\theta\}c_1}{\{(a^2+h_1^2b^2)-(a^2-h_1^2b^2)\cos 2\theta\}} - \frac{\{(a-h_2b)-(a+h_2b)\cos 2\theta\}c_2}{\{(a^2+h_2^2b^2)-(a^2-h_2^2b^2)\cos 2\theta\}}\right]$$

$$\overset{c}{p}_{xy} = \frac{-1}{(h_1-h_2)d}\left[\frac{c_1h_1(a+h_1b)\sin 2\theta}{\{(a^2+h_1^2b^2)-(a^2-h_1^2b^2)\cos 2\theta\}} - \frac{c_2h_2(a+h_2b)\sin 2\theta}{\{(a^2+h_2^2b^2)-(a^2-h_2^2b^2)\cos 2\theta\}}\right]$$

(298)

Using now the equation (71) and simplifying in the usual manner and noting that $\cos(x,n) = dy/ds$, $-\cos(y,n) = dx/ds$,

we have for the strain energy in the matrix

$$W_m = \frac{1}{4}\int_0^{2\pi}\Big[p^c_{xy}(\epsilon_2 b^2+\epsilon_1 a^2)\sin 2\theta - \epsilon_1 a b\, p^c_{xx}(1+\cos 2\theta) - \epsilon_2 a b\, p^c_{yy}(1-\cos 2\theta)\Big]d\theta .$$

Setting the values of p^c_{xx} , p^c_{yy} and p^c_{xy} from equations (298) in this and evaluating the integrals we shall obtain

$$W_m = \frac{-\pi}{2d}\Big[\epsilon_1^2 a^2 \alpha_{22}(h_1+h_2) + 2\,ab\,\epsilon_1\epsilon_2(\alpha_{22}+\alpha_{12}h_1h_2) + \epsilon_2^2 b^2 \alpha_{11} h_1h_2(h_1+h_2)\Big] \tag{299}$$

The total strain energy of the inclusion and matrix, by equations (290) and (299), is

$$W = \frac{\pi ab}{2}\left[\frac{(\epsilon_2-\delta_b)^2\beta_{11} + (\epsilon_1-\delta_a)^2\beta_{22} - 2(\epsilon_1-\delta_a)(\epsilon_2-\delta_b)\beta_{12}}{\beta_{11}\beta_{22}-\beta_{12}^2}\right]$$
$$+\frac{\pi}{-2d}\Big[\epsilon_1^2 a^2\alpha_{22}(h_1+h_2) + 2\,ab\,\epsilon_1\epsilon_2(\alpha_{22}+\alpha_{12}h_1h_2) + \epsilon_2^2 b^2\alpha_{11}h_1h_2(h_1+h_2)\Big] \tag{300}$$

Minimising W with respect to ϵ_1 and ϵ_2 gives

$$\epsilon_1 = \frac{\Big[\{\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)-d\}a + b\,\beta_{22}\alpha_{11}h_1h_2(h_1+h_2)\Big]b\delta_a - \{a\,\beta_{11}(\alpha_{22}+\alpha_{12}h_1h_2) + b\,\beta_{12}\alpha_{11}h_1h_2(h_1+h_2)\}b\delta_b}{\Big[ab\{h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2)-d\} + 2ab(\alpha_{22}+\alpha_{12}h_1h_2)\beta_{12} + a^2\beta_{11}\alpha_{22}(h_1+h_2) + b^2\beta_{22}\alpha_{11}h_1h_2(h_1+h_2)\Big]},$$

$$\epsilon_2 = \frac{\Big[\{\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)-d\}b + a\,\beta_{11}\alpha_{22}(h_1+h_2)\Big]a\delta_b - \{b\,\beta_{22}(\alpha_{22}+\alpha_{12}h_1h_2) + a\,\beta_{12}\alpha_{22}(h_1+h_2)\}a\delta_a}{ab\{h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2)-d\} + 2ab(\alpha_{22}+\alpha_{12}h_1h_2)\beta_{12} + a^2\beta_{11}\alpha_{22}(h_1+h_2) + b^2\beta_{22}\alpha_{11}h_1h_2(h_1+h_2)} . \tag{301}$$

Setting these values of ϵ_1, ϵ_2 in equations (287) and (288), the elastic field in the inclusion are determined. As regards the matrix we first determine the constants C_1 and C_2 in equations (297). These are

$$C_1 = \frac{-abh_1 d\left[\{a(\alpha_{22}-\alpha_{12}h_2^2)+a\beta_{12}h_2^2+b\beta_{22}h_2\}\delta_a + h_2\{b(\alpha_{12}-\alpha_{11}h_2^2)-b\beta_{12}-a\beta_{11}h_2\}\delta_b\right]}{ab\{h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2)-d\}+2ab(\alpha_{22}+\alpha_{12}h_1h_2)\beta_{12} + a^2\beta_{11}\alpha_{22}(h_1+h_2)+b^2\beta_{22}\alpha_{11}h_1h_2(h_1+h_2)},$$

$$C_2 = \frac{-abh_2 d\left[\{a(\alpha_{22}-\alpha_{12}h_1^2)+a\beta_{12}h_1^2+b\beta_{22}h_1\}\delta_a + h_1\{b(\alpha_{12}-\alpha_{11}h_1^2)-b\beta_{12}-a\beta_{11}h_1\}\delta_b\right]}{ab\{h_1h_2(\beta_{11}\beta_{22}-\beta_{12}^2)-d\}+2ab(\alpha_{22}+\alpha_{12}h_1h_2)\beta_{12} + a^2\beta_{11}\alpha_{22}(h_1+h_2)+b^2\beta_{22}\alpha_{11}h_1h_2(h_1+h_2)}. \tag{302}$$

Substituting these values of C_1 and C_2 in (296) and making use of (293), we determine $\varphi(z_1)$ and $\psi(z_2)$. The stress and displacement field in the matrix may then be evaluated by (248) and (249) respectively. Strains may be determined either by Hooke's law or strain-displacement relations.

The values of the normal and shear stresses at the common boundary of the matrix and inclusion are evaluated in the usual manner by making use of equations (117) and the trigonometrical relations (120)

$$\cos 2\beta = \frac{-\{(a^2-b^2)-(a^2+b^2)\cos 2\theta\}}{\{(a^2+b^2)-(a^2-b^2)\cos 2\theta\}}, \quad \sin 2\beta = \frac{-2ab\sin 2\theta}{\{(a^2+b^2)-(a^2-b^2)\cos 2\theta\}},$$

where β is the angular orientation of the outward drawn normal with respect to the x-axis. These are given by

$$P_{nn}^{L} = p_{nn}^{L} = \frac{\{(b+ah_1)c_1 - (b+ah_2)c_2\} - \{(b-ah_1)c_1 - (b-ah_2)c_2\}\cos 2\beta}{2ab(h_1-h_2)d},$$

$$P_{ns}^{L} = p_{ns}^{L} = \frac{\{(b-ah_1)c_1 - (b-ah_2)c_2\}\sin 2\beta}{2ab(h_1-h_2)d},$$

$$P_{ss}^{L} = \frac{\{(b+ah_1)c_1 - (b+ah_2)c_2\} + \{(b-ah_1)c_1 - (b-ah_2)c_2\}\cos 2\beta}{2ab(h_1-h_2)d},$$

$$p_{ss}^{L} = P_{ss}^{L} - \frac{2}{ab(h_1-h_2)d}\left[\frac{h_1(a+h_1b)c_1}{\{(1+h_1^2)-(1-h_1^2)\cos 2\beta\}} - \frac{h_2(a+h_2b)c_2}{\{(1+h_2^2)-(1-h_2^2)\cos 2\beta\}}\right].$$

(303)

Equations (303) establishes the continuity of the normal and shear stresses.

CHAPTER XXI

Elliptic Inclusion Problem in Orthotropic Elasticity
Shear Strain

In this chapter we consider the case when the orthotropic elliptic inclusion of semi-axes a and b tends to attain a free-surface which may be obtained by giving its boundary a displacement $\gamma_a y$, $\gamma_b x$ parallel to x and y-axes. Let us assume that the possible equilibrium boundary is an ellipse obtained by giving a displacement

$$u_x = \gamma_1 y \,, \qquad u_y = \gamma_2 x \tag{304}$$

to the inner boundary of the matrix, where γ_1 and γ_2 are unknown parameters to be determined.

For the inclusion, the boundary elastic displacement components are $(\gamma_1 - \gamma_a) y$, $(\gamma_2 - \gamma_b) x$. The displacement field in the inclusion, as already stated previously, may be taken to be

$$U_x = (\gamma_1 - \gamma_a) y \,, \qquad U_y = (\gamma_2 - \gamma_b) x \,. \tag{305}$$

whence the strains are given by

$$\varepsilon_{xx} = 0, \quad \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\gamma_1 + \gamma_2 - \gamma_a - \gamma_b\right) \tag{306}$$

and the corresponding stresses are

$$P_{xx} = 0, \quad P_{yy} = 0, \quad P_{xy} = \frac{\gamma_1 + \gamma_2 - \gamma_a - \gamma_b}{\beta_{66}} \tag{307}$$

Here again we use the constants β_{11}, β_{22}, β_{12}, β_{66} for the elastic properties of the inclusion material. By using the formula (40) the strain energy in the inclusion may be seen to be

$$W_I = \frac{\pi ab}{\beta_{cc}} \left\{ (\gamma_1+\gamma_2) - (\gamma_a+\gamma_b) \right\}^2 . \tag{308}$$

For the matrix the region outside the elliptic hole is mapped into a circle of unit radius $|\zeta| \leq 1$ in the ζ plane by the mapping function (292). We use the same notations for the mapping functions, boundary value of ζ, and complex potential functions etc. as we have used in the previous chapter. The boundary displacement components at its inner boundary are given by (304). Expressing x, y in terms of z, $\bar{z}$, setting z, $\bar{z}$ in terms of $\omega(\zeta)$ and $\overline{\omega(\zeta)}$ by equations (293), putting $\zeta=\sigma$ at the boundary and using the equations (255), we get

$$2\,\mathrm{Re}\left\{ m_1 \mathcal{F}(\sigma) + m_2 \mathcal{G}(\sigma) \right\} = \frac{i\gamma_1 b}{2}\left(\sigma - \frac{1}{\sigma}\right),$$

$$2\,\mathrm{Re}\left\{ n_1 \mathcal{F}(\sigma) + n_2 \mathcal{G}(\sigma) \right\} = \frac{\gamma_2 a}{2}\left(\sigma + \frac{1}{\sigma}\right) \tag{309}$$

Multiplying both sides of equations (309) by $(\sigma+\zeta)d\sigma/2\pi i(\sigma-\zeta)\sigma$, and using the Schwartz formula (256), we obtain

$$\mathcal{F}(\zeta) = \frac{i n_2 \gamma_1 b - m_2 \gamma_2 a}{2(m_1 n_2 - m_2 n_1)}\,\zeta , \quad \mathcal{G}(\zeta) = -\frac{i n_1 \gamma_1 b - m_1 \gamma_2 a}{2(m_1 n_2 - m_2 n_1)} . \tag{310}$$

Substituting the values of m_1, m_2, n_2, n_1 from equations (265) in equations (310) and simplifying, we shall get

$$\mathcal{F}(\zeta) = \frac{i c_1' \zeta}{2(h_1 - h_2)d} , \quad \mathcal{G}(\zeta) = -\frac{i c_2' \zeta}{2(h_1 - h_2)d} , \tag{311}$$

where

$$c_1' = h_1\{(\alpha_{22}-\alpha_{12}h_2^2)\, b\gamma_1 - (\alpha_{12}-\alpha_{11}h_2^2)\, h_2 a\gamma_2\},$$

$$c_2' = h_2\{(\alpha_{22}-\alpha_{12}h_1^2)\, b\gamma_1 - (\alpha_{12}-\alpha_{11}h_1^2)\, h_1 a\gamma_2\},$$

$$d = 2\alpha_{12}\alpha_{22} + \alpha_{12}^2 h_1 h_2 - \alpha_{11}\alpha_{22}(h_1^2 + h_1 h_2 + h_2^2) \,. \tag{312}$$

It may be seen that the expression for d is the same as in equations (297).

Transforming from ζ to z_1, z_2 planes with the help of the relations $z_1 = \omega_1(\zeta)$, and $z_2 = \omega_2(\zeta)$ given in equations (293), we evaluate $\varphi'(z_1)$ and $\psi'(z_2)$ and obtain

$$\varphi'(z_1) = \frac{i c_1'}{(h_1-h_2)d}\left\{\frac{\zeta^2}{(a-h_1 b)\zeta^2 - (a+h_1 b)}\right\},$$

$$\psi'(z_2) = \frac{-i c_2'}{(h_1-h_2)d}\left\{\frac{\zeta^2}{(a-h_2 b)\zeta^2 - (a+h_2 b)}\right\}. \tag{313}$$

The stress field in the matrix may be obtained from the relations (248) and (313). The stresses at the boundary of the hole may then be obtained. These are given by

$$p_{xx}^c = \frac{-\sin 2\theta}{(h_1-h_2)d}\left[\frac{c_1' h_1^2(a+h_1 b)}{\{(a^2+h_1^2 b^2)-(a^2-h_1^2 b^2)\cos 2\theta\}} - \frac{c_2' h_2^2(a+h_2 b)}{\{(a^2+h_2^2 b^2)-(a^2-h_2^2 b^2)\cos 2\theta\}}\right],$$

$$p_{yy}^c = \frac{\sin 2\theta}{(h_1-h_2)d}\left[\frac{c_1'(a+h_2 b)}{\{(a^2+h_1^2 b^2)-(a^2-h_1^2 b^2)\cos 2\theta\}} - \frac{c_2'(a+h_2 b)}{\{(a^2+h_2^2 b^2)-(a^2-h_2^2 b^2)\cos 2\theta\}}\right]$$

$$p_{xy}^{c} = \frac{1}{(h_1-h_2)d}\left[\frac{c_1' h_1\{(a-h_1 b)-(a+h_1 b)\cos 2\theta\}}{\{(a^2+h_1^2 b^2)-(a^2-h_1^2 b^2)\cos 2\theta\}} - \frac{c_2' h_2\{(a-h_2 b)-(a+h_2 b)\cos 2\theta\}}{\{a^2+h_2^2 b^2)-(a^2-h_2^2 b^2)\cos 2\theta\}}\right]. \tag{314}$$

By Clapeyron's theorem we get for the strain energy in the matrix

$$W_m = \frac{1}{4}\int\left[p_{xx}^{c} b^2 \gamma_1 \sin 2\theta + p_{yy}^{c} a^2 \gamma_2 \sin 2\theta - p_{xy}^{c} ab\{(\gamma_1+\gamma_2)-(\gamma_1-\gamma_2)\cos 2\theta\}\right] d\theta$$

$$= \frac{\pi}{2d}\left\{2\gamma_1\gamma_2 ab(\alpha_{12} h_1 h_2 + \alpha_{22}) - \gamma_1^2 b^2 \alpha_{22}(h_1+h_2) - \gamma_2^2 a^2 \alpha_{11} h_1 h_2 (h_1+h_2)\right\}, \tag{315}$$

where some of the integrals given in appendix III have been used.

The total strain energy W is given by

$$W = W_I + W_m = \frac{\pi ab}{2\beta_{66}}(\gamma_1+\gamma_2-\gamma_a-\gamma_b)^2$$

$$+\frac{\pi}{2d}\left\{2\gamma_1\gamma_2 ab(\alpha_{12} h_1 h_2 + \alpha_{22}) - \gamma_1^2 b^2 \alpha_{22}(h_1+h_2) - \gamma_2^2 a^2 \alpha_{11} h_1 h_2 (h_1+h_2)\right\}. \tag{316}$$

Setting $\partial W/\partial \gamma_1 = 0, \partial W/\partial \gamma_2 = 0$ for the extremum value of W and solving the equations, we have

$$\gamma_1 = \frac{a(\gamma_a+\gamma_b)\{a\alpha_{11} h_1 h_2 (h_1+h_2) + b(\alpha_{12} h_1 h_2 + \alpha_{22})\}}{a^2\alpha_{11} h_1 h_2 (h_1+h_2) + b^2\alpha_{22}(h_1+h_2) + 2ab(\alpha_{12} h_1 h_2 + \alpha_{22}) + ab\, h_1 h_2 \beta_{66}},$$

$$\gamma_2 = \frac{b(\gamma_a+\gamma_b)\}b\alpha_{22}(h_1+h_2) + a(\alpha_{12} h_1 h_2 + \alpha_{22})\}}{a^2\alpha_{11} h_1 h_2 (h_1+h_2) + b^2\alpha_{22}(h_1+h_2) + 2ab(\alpha_{12} h_1 h_2 + \alpha_{22}) + ab\, h_1 h_2 \beta_{66}}. \tag{317}$$

It may be verified that these give the minimum value for W.

From these values of γ_1 and γ_2, the elastic field both in the inclusion and matrix may be determined by following the

procedure indicated in the previous chapter. In particular, the normal and shear stresses at the equilibrium interface are given by

$$P^{b}_{nn} = p^{b}_{nn} = \frac{-h_1 h_2 a b(\gamma_a + \gamma_b)\sin 2\beta}{a^2\alpha_{11} h_1 h_2(h_1+h_2) + b^2\alpha_{22}(h_1+h_2) + 2ab(\alpha_{12}h_1h_2+\alpha_{22}) + ab\,h_1h_2\beta_{66}},$$

$$P^{b}_{ns} = p^{b}_{ns} = \frac{-h_1 h_2 a b(\gamma_a + \gamma_b)\cos 2\beta}{a^2\alpha_{11} h_1 h_2(h_1+h_2) + b^2\alpha_{22}(h_1+h_2) + 2ab(\alpha_{12}h_1h_2+\alpha_{22}) + ab\,h_1h_2\beta_{66}} \tag{318}$$

which substantiates our basic assumption for the equilibrium boundary given by (304). The hoop stress in the inclusion is

$$P^{b}_{ss} = -P^{b}_{nn} = \frac{h_1 h_2 a b(\gamma_a + \gamma_b)\sin 2\beta}{a^2\alpha_{11} h_1 h_2(h_1+h_2) + b^2\alpha_{22}(h_1+h_2) + 2ab(\alpha_{12}h_1h_2+\alpha_{22}) + ab\,h_1h_2\beta_{66}}, \tag{319}$$

and in the matrix, it is

$$p^{b}_{ss} = P^{b}_{ss} + \frac{2\sin 2\beta}{ab(h_1-h_2)d}\left[\frac{c_1' h_1(b-ah_1)}{\{(1+h_1^2)-(1-h_1^2)\cos 2\beta\}} - \frac{c_2' h_2(b-ah_2)}{\{(1+h_2^2)-(1-h_2^2)\cos 2\beta\}}\right] \tag{320}$$

where

$$c_1' = \frac{-h_1 h_2(a+bh_2)ab(\gamma_a+\gamma_b)d}{a^2\alpha_{11} h_1 h_2(h_1+h_2) + b^2\alpha_{22}(h_1+h_2) + 2ab(\alpha_{12}h_1h_2+\alpha_{22}) + ab\,h_1h_2\beta_{66}},$$

$$c_2' = \frac{-h_1 h_2(a+bh_1)ab(\gamma_a+\gamma_b)d}{a^2\alpha_{11} h_1 h_2(h_1+h_2) + b^2\alpha_{22}(h_1+h_2) + 2ab(\alpha_{12}h_1h_2+\alpha_{22}) + ab\,h_1h_2\beta_{66}}. \tag{321}$$

It may be seen from equations (318) to (320) that the values of the normal and hoop stresses are zero at the interface

both in the inclusion and matrix for $\beta = 0$, $\pi/2$, π and $3\pi/2$. Further the hoop stresses in the inclusion bear the same sign in the first and third quadrants of the ellipse and the opposite sign in the second and fourth quadrants. In other words, if they are tensile in the first and third quadrants, they will be compressive in second and fourth quadrants. They will be of opposite sense in the matrix.

The strain energy in the inclusion may be obtained in terms of known quantities by substituting the values of γ_1 and γ_2 in (308). Similarly the strain energy in the matrix may be obtained by setting the values of γ_1 and γ_2 in (315). As regards the continuity of the displacement at the interface, we have already assumed that the net displacements there are the same.

It may be added here that if the inclusion tends to undergo a non-elastic displacement with components $(\delta_a x + \gamma_a y)$, $(\delta_b y + \gamma_b x)$ parallel to the coordinate axes, then in the equilibrium position the net displacement of the boundary of the ellipse $x^2/a^2 + y^2/b^2 = 1$ would have the components $(\epsilon_1 x + \gamma_1 y)$, $(\epsilon_2 y + \gamma_2 x)$ where ϵ_1 , ϵ_2 , γ_1 and γ_2 are given by equations (301) and (317). The stresses and strains can be obtained by superposition. It may be remarked here that all of this analysis is correct for negative values of δ_a, δ_b, γ_a, γ_b also if continuity of the inclusion and matrix is maintained.

CHAPTER XXII

On Orthotropic Elliptic Inclusion

In this chapter we derive certain results from the analysis in the last two chapters.

Taking first the principal strain case the substitution of the values of parameters ϵ_1 , ϵ_2 obtained in equations (301), in the appropriate expression determines the elastic field in the inclusion and matrix. It is of interest to know how the results simplify for some particular cases.

Taking the materials of the inclusion and matrix to be orthotropic having the same elastic constants (and therefore setting $\beta_{11} = \alpha_{11}$, $\beta_{22} = \alpha_{22}$, $\beta_{12} = \alpha_{12}$) the expressions

$$\epsilon_1 = \frac{\begin{array}{l}\left[\alpha_{22}\{\alpha_{11}(h_1^2 + h_1 h_2 + h_2^2) - \alpha_{12}\} a + b\,\alpha_{11}\alpha_{22} h_1 h_2 (h_1 + h_2)\right] b\delta_a \\ \qquad - \{a\,\alpha_{11}(\alpha_{22} + \alpha_{12} h_1 h_2) + b\,\alpha_{12}\alpha_{11}(h_1 + h_2) h_1 h_2\} b\delta_b\end{array}}{\alpha_{11}\alpha_{22}(h_1 + h_2)\{a^2 + ab(h_1 + h_2) + b^2 h_1 h_2\}},$$

$$\epsilon_2 = \frac{\begin{array}{l}\left[\alpha_{22}\{\alpha_{11}(h_1^2 + h_1 h_2 + h_2^2) - \alpha_{12}\} b + a\,\alpha_{11}\alpha_{22} (h_1 + h_2)\right] a\delta_b \\ \qquad - \{b\,\alpha_{22}(\alpha_{22} + \alpha_{12} h_1 h_2) + a\,\alpha_{12}\alpha_{22}(h_1 + h_2)\} a\delta_a\end{array}}{\alpha_{11}\alpha_{22}(h_1 + h_2)\{a^2 + ab(h_1 + h_2) + b^2 h_1 h_2\}}. \tag{322}$$

It may be repeated that the intersections of the planes of elastic symmetry are the same both for the inclusion and matrix, and further the nomenclature of the axes is also the same.

As a next case, if the inclusion is of orthotropic material and the matrix of cubic, we have only to set $\alpha_{11} = \alpha_{22}$ in (301). Finally if the matrix is isotropic in place of cubic then

$h_1 = h_2 = 1$ and ϵ_1, ϵ_2 reduce to

$$\epsilon_1 = \frac{\left[\{\beta_{12}(\alpha_{11}+\alpha_{12}) - d\} a + 2b\beta_{22}\alpha_{11}\right] b\delta_a - \{a\beta_{11}(\alpha_{11}+\alpha_{12}) + 2b\beta_{12}\alpha_{11}\} b\delta_b}{ab\{(\beta_{11}\beta_{22} - \beta_{12}^2) - d\} + 2ab(\alpha_{11}+\alpha_{12})\beta_{12} + 2\alpha_{11}(a^2\beta_{11} + b^2\beta_{22})},$$

$$\epsilon_2 = \frac{\left[\{\beta_{12}(\alpha_{11}+\alpha_{12}) - d\} b + 2a\beta_{11}\alpha_{11}\right] a\delta - \{b\beta(\alpha+\alpha) + 2a\beta\alpha\} a\delta}{ab\{(\beta_{11}\beta_{22} - \beta_{12}^2) - d\} + 2ab(\alpha_{11}+\alpha_{12})\beta_{12} + 2\alpha_{11}(a^2\beta_{11} + b^2\beta_{22})}, \qquad (323)$$

where $d = 2\alpha_{12}\alpha_{11} + \alpha_{12}^2 - 3\alpha_{11}^2$, $\alpha_{11} = \alpha_{22} = \dfrac{1-\nu^2}{E}$, $\alpha_{12} = \dfrac{-\nu(1+\nu)}{E}$.

On the other hand, if the matrix is orthotropic, inclusion being cubic, we again set $\beta_{11} = \beta_{22}$. But if the inclusion is isotropic, matrix remaining orthotropic, (the values of $\beta_{11}, .., .., ..,$ may be put in terms of Young's modulus and Poisson's ratio in the same manner as $\alpha_{11}, .., .., ..,$ are put), ϵ_1, ϵ_2 are given by

$$\epsilon_1 = \frac{\left[\{\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2) - d\} a + b\beta_{11}\alpha_{11}h_1h_2(h_1+h_2)\right] b\delta_a - \{a\beta_{11}(\alpha_{22}+\alpha_{12}h_1h_2) + b\beta_{12}\alpha_{11}h_1h_2(h_1+h_2)\} b\delta_b}{ab\{h_1h_2(\beta_{11}^2 - \beta_{12}^2) - d\} + 2ab(\alpha_{22}+\alpha_{12}h_1h_2)\beta_{12} + (a^2\alpha_{22} + b^2\alpha_{11}h_1h_2)(h_1+h_2)\beta_{11}},$$

$$\epsilon_2 = \frac{\left[\{\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2) - d\} b + a\beta_{11}\alpha_{22}(h_1+h_2)\right] a\delta_b - \{b\beta_{11}(\alpha_{22}+\alpha_{12}h_1h_2) + a\beta_{12}\alpha_{22}(h_1+h_2)\} a\delta_a}{ab\{h_1h_2(\beta_{11}^2 - \beta_{12}^2) - d\} + 2ab(\alpha_{22}+\alpha_{12}h_1h_2)\beta_{12} + (a^2\alpha_{22} + b^2\alpha_{11}h_1h_2)(h_1+h_2)\beta_{11}} \qquad (324)$$

It may bear mention that the essential difference between cubic and isotropic plane problems would be as follows. For cubic materials, we set $\beta_{11} = \beta_{22}$ and have to evaluate the values of h_1, h_2 from the characteristic equation (240). In isotropic materials too, we set $\beta_{11} = \beta_{22}$, but $h_1 = h_2 = 1$

and β_{11} would be related to β_{66} and β_{12}.

As a next case if the matrix is rigid i.e. $\alpha_{11} = \alpha_{22} = \alpha_{12} = \alpha_{66} = 0$ or if there is a cavity i.e. $\beta_{11} = \beta_{22} = \beta_{12} = \infty$ then $\epsilon_1 = \epsilon_2 = 0$. Similarly if the inclusion is rigid, $\beta_{11} = \beta_{22} = \beta_{12} = 0$ it may be seen that $\epsilon_1 = \delta_a$, $\epsilon_2 = \delta_b$ which is obvious on physical grounds.

The results obtained in (322), (323) and (324) may be further specialised for the circular case $a = b$. Corresponding to these equations we obtain for the circular inclusion

$$\epsilon_1 = \frac{\left[\alpha_{11}\{(h_1+h_2)^2 + h_1h_2(h_1+h_2-1)\} - \alpha_{12}\right]\alpha_{22}\delta_a - \{\alpha_{22} + \alpha_{12}h_1h_2(h_1+h_2+1)\}\alpha_{11}\delta_b}{\alpha_{11}\alpha_{22}(h_1+h_2)(1+h_1+h_2+h_1h_2)},$$

$$\epsilon_2 = \frac{\left[\alpha_{11}\{(h_1+h_2)^2 + (h_1+h_2-h_1h_2)\} - \alpha_{12}\right]\alpha_{22}\delta_b - \{\alpha_{22} + \alpha_{12}(h_1+h_2+h_1h_2)\}\alpha_{22}\delta_a}{\alpha_{11}\alpha_{22}(h_1+h_2)(1+h_1+h_2+h_1h_2)};$$

$$\epsilon_1 = \frac{\{\beta_{12}(\alpha_{11}+\alpha_{12}) - d + 2\alpha_{11}\beta_{22}\}\delta_a - \{\beta_{11}(\alpha_{11}+\alpha_{12}) + 2\beta_{12}\alpha_{11}\}\delta_b}{\beta_{11}\beta_{22} - \beta_{12}^2 - d + 2\beta_{12}(\alpha_{11}+\alpha_{12}) + 2\alpha_{11}(\beta_{11}+\beta_{22})},$$

$$\epsilon_2 = \frac{\{\beta_{12}(\alpha_{11}+\alpha_{12}) - d + 2\alpha_{11}\beta_{11}\}\delta_b - \{\beta_{22}(\alpha_{11}+\alpha_{12}) + 2\beta_{12}\alpha_{11}\}\delta_a}{\beta_{11}\beta_{22} - \beta_{12}^2 - d + 2\beta_{12}(\alpha_{11}+\alpha_{12}) + 2\alpha_{11}(\beta_{11}+\beta_{22})};$$

$$\epsilon_1 = \frac{\{\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2) - d + \beta_{11}\alpha_{11}h_1h_2(h_1+h_2)\}\delta_a - \{\beta_{11}(\alpha_{22}+\alpha_{12}h_1h_2) + \beta_{12}\alpha_{11}h_1h_2(h_1+h_2)\}\delta_b}{h_1h_2(\beta_{11}^2-\beta_{12}^2) - d + 2\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2) + (\alpha_{22}+\alpha_{11}h_1h_2)(h_1+h_2)\beta_{11}},$$

$$\epsilon_2 = \frac{\{\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2) - d + \beta_{11}\alpha_{22}(h_1+h_2)\}\delta_b - \{\beta_{11}(\alpha_{22}+\alpha_{12}h_1h_2) + \beta_{12}\alpha_{22}(h_1+h_2)\}\delta_a}{h_1h_2(\beta_{11}^2-\beta_{12}^2) - d + 2\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2) + (\alpha_{22}+\alpha_{11}h_1h_2)(h_1+h_2)\beta_{11}}.$$

(325)

As an example for one of these, we might cite the following which occurs quite frequently in practice. Consider a metallic nail supposed to be isotropic, which has been drawn into a piece of wood a common orthotropic material. The coordinate axes have already been chosen in such a manner that real parts of the roots of the characteristic equation are zero. The resulting elastic field may be derived from the last of equations (325).

As a next example if a is large in comparison with b

$$\epsilon_1 = 0 \, , \qquad \epsilon_2 = \frac{\beta_{11}\delta_b - \beta_{12}\delta_a}{\beta_{11}} \, . \tag{326}$$

On the other hand if b is large in comparison with a i.e. $a/b \to 0$

$$\epsilon_1 = \frac{\beta_{22}\delta_a - \beta_{12}\delta_b}{\beta_{22}} \, , \qquad \epsilon_2 = 0 \, . \tag{327}$$

The expressions in (326) and (327) differ because of the orthotropic character of elastic material.

As further examples one might consider the inclusion to be isotropic and matrix to be cubic or isotropic or vice-versa. When both the inclusion and the matrix are of the isotropic materials, we may derive the values of ϵ_1 , ϵ_2 by setting the values of $\alpha_{11}, \ldots, \beta_{11}, \ldots$ in terms of Young's moduli and Poisson's ratios already given above. For the elliptic case these are obtained from equations (301). They are identical with those in (113), which is a useful check on the results.

It is also interesting to see that the shear stresses in

principal strain transformation case are zero at $\beta = 0, \pi/2, \pi$ and $3\pi/2$ i.e., at the ends of the major and minor axes. The directions of the axes at these points are thus the principal stress directions. Further at the tips of the axes, the hoop stresses in the inclusion assume extremum values. The jump in the hoop stress or their ratio may be derived from the results in equations (302). It may also be added that for the stresses, the constants C_1, C_2 are to be evaluated. These may be determined from equations (301).

It may be observed that incidentally we have solved the problem of an elliptic hole under uniform pressure P in an infinite orthotropic elastic medium. This can be simply done by choosing δ_a, δ_b in such a manner that

$$C_1(b - a h_1) - (b - a h_2) C_2 = 0$$

$$\frac{C_1(b + a h_1) - (b + a h_2) C_2}{2ab(h_1 - h_2)d} = -P. \tag{328}$$

We can similarly deduce a few other interesting results.

As regards the shear strain transformation of the inclusion, one might derive results similar to those of principal strain. It is quite interesting to note that β_{66} is the only elastic constant pertaining to the inclusion material which occurs in the expressions for γ_1, γ_2. The constants β_{11}, β_{12}, β_{22} do not play any part. Therefore, the results (317) remain unaltered whether the inclusion is orthotropic, cubic or isotropic.

One has to simply set the correct value of β_{66} . Of course, in the statement it is implicit that the axes of the inclusion and matrix must be chosen in the way mentioned earlier.

If the matrix is isotropic,

$$\gamma_1 = \frac{a(\gamma_a+\gamma_b)\left\{2a\alpha_{11}+b(\alpha_{11}+\alpha_{12})\right\}}{2\alpha_{11}(a^2+ab+b^2)+(2\alpha_{12}+\beta_{66})ab},$$

$$\gamma_2 = \frac{b(\gamma_a+\gamma_b)\left\{2b\alpha_{11}+a(\alpha_{11}+\alpha_{12})\right\}}{2\alpha_{11}(a^2+ab+b^2)+(2\alpha_{12}+\beta_{66})ab}. \tag{329}$$

By setting $a=b$, we may get the results of the circular case. In this case

$$\gamma_1 = \gamma_2 = \frac{(\gamma_a+\gamma_b)(3\alpha_{11}+\alpha_{12})}{2(3\alpha_{11}+\alpha_{12})+\beta_{66}}.$$

The equilibrium boundary, in the case of principal strain is a simple affair. It would be an ellipse whose axes coincide with those of the hole, and are of lengths $2a(1+\epsilon_1)$ and $2b(1+\epsilon_2)$

In case of shear strain, however, the equilibrium boundary is again an ellipse whose parametric equations will be

$$x = a\cos\theta + \gamma_1 b\sin\theta, \quad y = b\sin\theta + \gamma_2 a\cos\theta. \tag{330}$$

The major axis of this ellipse will be inclined by a small angle α to the initial major axis, given by

$$\alpha = \frac{a^2\gamma_2 + b^2\gamma_1}{b^2-a^2}.$$

It may be noted that initially the major axis of the free surface of the inclusion was making an angle

$$\frac{a^2\gamma_b + b^2\gamma_a}{b^2 - a^2}$$

to the major axis of the hole.

The lengths of the major and minor axes of this ellipse will respectively be

$$2\left[\frac{a^2}{a^2-b^2}\left\{(a^2-b^2)(1-\gamma_1^2) + a^2(\gamma_1+\gamma_2)^2\right\}\right]^{\frac{1}{2}},$$

$$2\left[\frac{b^2}{a^2-b^2}\left\{(a^2-b^2)(1-\gamma_2^2) - b^2(\gamma_1+\gamma_2)^2\right\}\right]^{\frac{1}{2}},$$

which neglecting the second power also of γ_1 , γ_2 remain equal to $2a$ and $2b$. γ_1 , γ_2 occuring in (330) are given by (317).

As another example of shear strain, one may take $\delta_a = -\delta_b = \delta$. Note that in all the previous analysis we have no where assumed that δ_a , δ_b should only be positive. What has been assumed is the continuity of the material. The results of this type of shear strain may be derived from the case of principal strain by setting $\delta_a = -\delta_b = \delta$. It may be checked that in the general orthotropic case ϵ_1 will not be equal to ϵ_2 .

As already remarked, we can solve quite a few problems in orthotropic elasticity related to elliptic hole. Next one might consider the problem of an elliptic inclusion in stressed orthotropic medium. These problems are no more difficult than those dealt with in chapters XII to XV and one may obtain results,

by following a procedure similar to those followed in the chapters cited above.

It is gratifying to note that by the use of energy principles, more realistic problems can be solved in a relatively simpler manner than the point-force method. However, it may bear mention, that in principle there is no fundamental difference between the two. It only states that the effect of the point-force may be to bring the inclusion and matrix to a position which is in the minimum energy configuration.

PART III

CHAPTER XXIII

Numerical Method of Solving Laplace's Equation

The investigations of this and the next chapter were taken, as it was thought to solve some problems relating to inclusions subjected to twisting actions. This would involve the Laplace's equation in two-dimensions. Unfortunately modern computing aids were not available, and hence the preliminary work had to be carried out on a small desk calculating machine.

The solution of Laplace's equation in two-dimensions may be found by analytical methods when the boundaries are sufficiently simple and so also are the boundary conditions. It may also be found by electrical analogy methods. Another mathematical technique is the relaxation method developed by Southwell [18] , Allen [19] and others. This method is based upon finite differences. The procedure however, becomes combursome if the boundary is curved or if there are very large number of nodal points. However, a method based on Green's formula [20] may be formulated. This would involve the solution of integral equation. It appears, it would have the advantage of being programmed on a digital computer.

The solution of Laplace's equation when applied to a particular problem depends on the boundary conditions. The boundary conditions may be specified by the value of

1 the function on the boundary, or
2 the normal derivative of the function on the boundary, or
3 the function on a part of the boundary and its normal derivative on the remaining part.

These are respectively called as Dirchlet's, Neumann's and Cauchy's problems in the literature. All these conditions can be summarised by stating that $(a\, \partial f/\partial n) + bf = c$ on the boundary, where n is the outward drawn normal to the boundary. The following is a common theory of solving all the above problems.

The basic integral equation is derived from the Green's formula [20] , which states that if ϕ satisfies the Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{331}$$

in the domain D bounded by Γ , and ϕ has the value f on the boundary, then the expression

$$[f, f'] = \frac{1}{2\pi}\int_\Gamma f(s) \log' |s - P| \, ds - \frac{1}{2\pi}\int_\Gamma f'(s) \log |s - P| \, ds, \tag{332}$$

has the value

$\phi(P)$ for a pole anywhere in D ,

$f(p)$ for a pole just inside Γ ,

$\frac{1}{2} f(p)$ for a pole on Γ ,

0 for a pole just and everywhere outside Γ .

In expression (332) s denotes the current point on Γ , ds the arc length, P denotes the fixed point (pole) within or without D , a pole on Γ being denoted by p . $|s-P|$ is the numerical distance between s and P . It is obvious that $\frac{\partial}{\partial n} \log|s-P|$ simply means the normal derivative of $\log|s-P|$ at s .

This suggests a new approach to Dirchlet's and Neumann's problems. Thus, for the Dirchlet's problem, we calculate $f'(s)$ from the linear integral equation

$$\frac{1}{2\pi}\int f'(s)\log|s-P|\,ds = \frac{1}{2\pi}\int f(s)\log'|s-P|\,ds - \frac{1}{2}f(P), \tag{333}$$

whence the function (f, f') may be determined throughout D. This is the Fredholm equation of the first kind, with Kernel $\log|s-P|$. It may be proved that $f'(s)$ calculated from (333) satisfies the relation

$$\int_\Gamma f'(s)\,ds = 0. \tag{334}$$

Similarly to determine the harmonic function having given values $f'(s)$ on Γ, we calculate $f(s)$, apart from a constant, from the linear integral equation

$$\frac{1}{2\pi}\int f(s)\log'|s-P|\,ds = \frac{1}{2\pi}\int f'(s)\log|s-P|\,ds + \frac{1}{2}f(P). \tag{335}$$

This is a Fredholm equation of the second kind with kernel $\log'|s-P|$ and admits a solution only if the assigned values of $f'(s)$ satisfies the relation (334) [21].

The application of the new formulation of the integral equation involves the determination of the kernels $\log'|s-P|$ and $\log|s-P|$, which can be evaluated as explained below.

$$\begin{aligned}\log'|s-P| &= \frac{d}{dn}\log|s-P|,\\ &= \frac{\partial \log}{\partial x}|s-P|\frac{\partial x}{\partial n} + \frac{\partial \log}{\partial y}|s-P|\frac{\partial y}{\partial n},\\ &= \frac{(x-x_1)\cos\beta + (y-y_1)\sin\beta}{(x-x_1)^2+(y-y_1)^2},\end{aligned} \tag{336}$$

where (x, y) are the coordinates of s and (x_1, y_1) of P and hence fixed and β is the angle of inclination of the outward normal at s to x-axis.

The values of $\log'|s-P|$ can be checked by the application of Cauchy-Riemann equation for an analytical function, which yields

$$\oint \frac{d}{dn} \log|s-P| \, ds = \oint \frac{d\theta}{ds} \, ds ,$$
$$= 2\pi \text{ for } P \text{ inside } \Gamma ,$$
$$= \pi \text{ for } P \text{ on } \Gamma ,$$
$$= 0 \text{ for } P \text{ outside } \Gamma . \qquad (337)$$

$\log|s-P|$ can be got from any mathematical tables for all values of $|s-P|$ sufficiently greater than zero. But as $|s-P| \to 0$ it may be determined by taking the average of $\log|s-P|$ through an interval of $ds/2$ on either side of s along Γ. This is done as follows.

If the arc adjoining p is approximated to a straight line of length a, then

$$\log|s-P| = \frac{1}{a}\int_0^a \log s \, ds ,$$
$$= \frac{1}{a}\Big| s \log s - s \Big|_0^a$$
$$= \log a - 1 \qquad (338)$$

To get a better approximation, the arc adjoining p is approximated to the arc of a circle of radius a, subtending an angle 2α at its centre. In this case $ds = 2a\alpha$ we have, as

$$|s-P| \to 0, \ \log|s-P| = \frac{1}{a\alpha}\int_0^\alpha \log\left(a \sin\frac{\theta}{2}\right) a \, d\theta ,$$

$$= \frac{1}{\alpha}\left\{\alpha \log a + \int_0^\alpha \log \sin\frac{\theta}{2} \, d\theta\right\} = \frac{1}{\alpha}\left\{\alpha \log a + \alpha \log \sin\frac{\alpha}{2} - 2\int_0^{\alpha/2} \theta \cot\theta \, d\theta\right\} . \qquad (339)$$

$\int_0^{\alpha/2} \theta \cdot \cot\theta \, d\theta$ is evaluated by writing $\theta \cot\theta = \cos\theta \Big/ \frac{\sin\theta}{\theta}$ and expanding both the numerator and the denominator in terms of θ.

For small values of θ we have

$$\int_0^{\alpha/2} \theta \cot\theta \, d\theta = \int_0^{\alpha/2} \left(1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}\right) d\theta = \frac{\alpha}{2} - \left(\frac{\alpha}{2}\right)^3 \frac{1}{9} - \left(\frac{\alpha}{2}\right)^5 \frac{1}{225} .$$

$$\therefore \log|s - P| = \log a + \log \sin\frac{\alpha}{2} - 1 + \frac{\alpha^2}{36} + \frac{\alpha^4}{3600} ,$$

$$\text{as } |s - P| \to 0$$

when $a = 1$, as $|s-P| \to 0$, $\log|s-P| = \log\sin\frac{\alpha}{2} - 1 + \frac{\alpha^2}{36} + \frac{\alpha^4}{3600} = I$ (340)

The values of I for $\alpha = 1°$ to $10°$ and $\pi/12$, $\pi/16$, $\pi/24$, $\pi/30$, $\pi/36$, $\pi/8$ are tabulated below.

α	$\frac{1}{\alpha}\int \log \sin\frac{\theta}{2} \, d\theta$	α	$\frac{1}{\alpha}\int \log \sin\frac{\theta}{2} \, d\theta$
1°	- 5.04841	$\pi/8$	- 1.95621
2°	- 4.35524	$\pi/12$	- 2.34970
3°	- 3.94994	$\pi/16$	- 2.63329
4°	- 3.66246	$\pi/24$	- 3.03581
5°	- 3.43980	$\pi/30$	- 3.25867
6°	- 3.25867	$\pi/36$	- 3.43980
7°	- 3.10455		
8°	- 2.97151		
9°	- 2.85454		
10°	- 2.74992		

In most cases the approximation of the arc to an arc of a circle should suffice. However, for some simple cases e.g. ellipse, parabola etc., an analytical formulae correct to the desired degree of accuracy can be evaluated.

CHAPTER XXIV

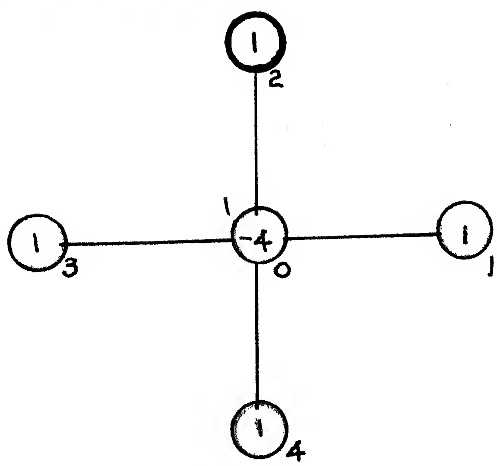
Torsion Problem of Rectangular Prism

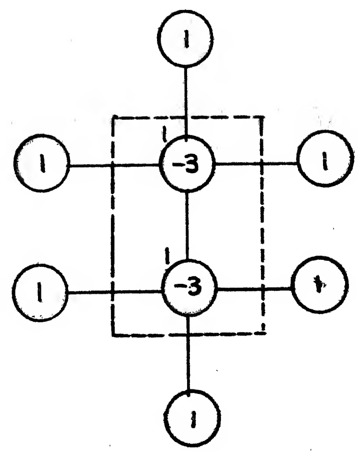
We now apply the method explained in the previous chapter for solving elastic torsion problem of a uniform isotropic bar of rectangular cross section. This section has been particularly chosen, as there is a sudden change in the direction of normal and is, therefore, likely to give more inaccurate result by this method. The analytical solution of this problem is known. Further this boundary is particularly simple for the relaxation case. We shall show that, checked with analytical solution, the results by the method gives comparable results with those obtained by the use of relaxation techniques.

We choose the dimensions of the section to be $a = 64$ parallel to x-axis and $b = 32$ parallel to y-axis. These values were taken simply to facilitate the subdivision of the boundary. The value of the torsion function on the boundary is given by

$\psi = \frac{1}{2}(x^2 + y^2)$ [8].

For obtaining the solution by the relaxation technique [18 , 19] the area of integration were divided into a uniform rectangular net work and the values of the functions at the nodes of this net work were determined.





As the boundary is straight and rectangular the standard relaxation tables shown above were used. Block relaxation was also adopted whenever it was feasible. A relaxation table as shown below was chosen. The values at the nodes were assumed after determining the values of the functions at the boundary. The residuals were calculated by

$$F_0 = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 - 4\Psi_0 .$$

128	136	160	200	256	328	416	520	640
173	180	205	235	285	351	451	506	
176	183	203	241	290	352	424	504	
175	183	205	241	291	353	426	506	585
208	213	235	267	318	365	425	487	
209	216	236	268	312	365	425	487	
212	215	235	268	313	369	428	491	544
229	232	256	278	324	380	422	475	
229	235	254	284	324	372	428	476	
227	234	253	284	325	374	427	481	520
232	239	260	286	326	373	428	470	
233	241	260	289	328	374	423	472	
234	240	259	289	328	376	427	476	512

(In the above table the lowest figures at a nodal point indicate the analytical values, the middle ones obtained by relaxation method, and the top most the values, by the method of this paper.)

Then the residuals were liquidated by the known relaxation technique and checked [18, 19].

These values were checked analytically by the application of the known solution [8]

$$\psi_{(x,y)} = \frac{b^2}{4} + \frac{x^2-y^2}{2} - \frac{8b^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\,\frac{\cosh K_n x}{\cosh\left(\frac{K_n a}{2}\right)}\,\cos K_n y\ ,$$

where

$$K_n = \frac{(2n+1)\pi}{b}\ .$$

These values are indicated in the above table itself, which indicate that the values obtained by relaxation technique compare well with those of analytical values.

Then this numerical method was applied for which the value of the normal derivative $d\psi/d\nu$ was taken from [8]

$$\frac{d\psi}{dx} = x - \frac{8b}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\,\frac{\sinh K_n x}{\cosh\left(\frac{K_n a}{2}\right)}\,\cos K_n y\ ,$$

$$\frac{d\psi}{dy} = -y + \frac{8b}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\,\frac{\sinh K_n x}{\cosh\left(\frac{K_n a}{2}\right)}\,\sin K_n y\ ,$$

and are indicated in the next page. Using these values of $d\psi/d\nu$, the values of the function at other points were determined and are indicated in the table. A comparison of the three results indicate that the proposed numerical method is as reliable as the relaxation technique with added advantages.

(x,y)	$d\psi/d\nu$	(x,y)	$d\psi/d\nu$
(32,2)	8.650	(22,16)	- 7.745
(32,6)	9.650	(18,16)	- 9.023
(32,10)	15.000	(14,16)	-11.330
(32,14)	25.340	(10,16)	-12.620
(30,16)	10.140	(6,16)	-13.390
(26,16)	-0.830	(2,16)	-13.745

The integral equation involving the values of the function and its normal derivative on the boundary, deduced by the Green's formula, enables us to know the normal derivative or the function whichever is not given. Then the same formula will give us the value of the function anywhere with in the boundary. (It may be remarked here that relaxation method will give us the value only at certain points which are the nodes of the net work). The method is applicable even when the boundary conditions are specified by the values of the function on a part of the boundary and of its normal derivative on the remaining part. Further if the boundary values consist of ϕ_x and ϕ_y, these enables us to know ϕ_ν and $d\phi/d\nu$.

For a given Γ the table can be prepared for the various values of $\log|s-P|$ and a chosen ds and then the value of ϕ and $d\phi/d\nu$ can be substituted to obtain the value of ϕ at any other point, i.e., for a given boundary, the problem need not be commenced afresh for different values of ϕ and $d\phi/d\nu$.

$$\epsilon = \frac{3K_1 \delta}{3K_1 + 4\mu} = \frac{E_1 (1+\nu)\delta}{E_1(1+\nu)+2E(1-2\nu_1)} .$$

It can be seen that this result is identical with that obtained by Nabarro as in chapter I, when the materials of inclusion and matrix are the same.

APPENDIX II

Use of Complex Variable Technique in Circular Inclusion Problem

The notations used are the same as those used in chapter V. The displacement components, strains, stresses and strain energy in the inclusion are

$$U_x = (\epsilon - \delta)x\,, \quad U_y = (\epsilon - \delta)y\,,$$

$$\mathcal{E}_{xx} = (\epsilon - \delta)\,, \quad \mathcal{E}_{yy} = (\epsilon - \delta)\,, \mathcal{E}_{xy} = 0\,,$$

$$P_{xx} = P_{yy} = 2(\lambda_1 + \mu_1)(\epsilon - \delta)\,, \; P_{xy} = 0\,,$$

$$W_I = 2\pi a^2 (\lambda_1 + \mu_1)(\epsilon - \delta)^2.$$

In order to determine the elastic field in the infinite region with a circular hole of radius a, when its boundary undergoes a displacement whose components are $u_x = \epsilon x$, $u_y = \epsilon y$ we map the region outside the circle in z-plane to a region within the unit circle $|\zeta| = 1$ in the ζ-plane by the mapping function $z = a/\zeta$. Making use of the boundary conditions and equation (87), the functions F and G are determined by the equation

$$kF(\sigma) + \frac{1}{\sigma}\overline{F'(\sigma)\sigma^2} - \overline{G(\sigma)} = \frac{2a\mu}{\sigma}\epsilon.$$

Solving the above equation by the method explained in chapter VI, we get

$$F(\zeta) = 0\,, G(\zeta) = -2a\mu\epsilon\zeta\,.$$

The stresses, therefore, are given by

$$p_{xx} + p_{yy} = 0\,,$$

$$p_{yy} - p_{xx} + 2i\, p_{xy} = 4\mu\epsilon\zeta^2.$$

By setting $\zeta = \sigma = e^{i\theta}$, we obtain the boundary values of the stresses to be

$$p^c_{xx} = -2\mu\epsilon \cos 2\theta, \quad p^c_{yy} = 2\mu\epsilon \cos 2\theta, \quad p^c_{xy} = 2\mu\epsilon \sin 2\theta.$$

Making use of Clapeyron's theorem, we get the strain energy in the matrix. It is given by

$$W_m = 2\pi a^2 \mu \epsilon^2.$$

The total strain energy of the system is

$$W = W_I + W_m = 2\pi a^2 \{ (\lambda_1 + \mu_1)(\epsilon - \delta)^2 + \mu\epsilon^2 \} .$$

Minimising this, we get

$$\epsilon = \frac{(\lambda_1 + \mu_1)\delta}{\lambda_1 + \mu_1 + \mu} ,$$

which is identical with (65) in chapter V .

APPENDIX III

Some Integrals

$$\int \frac{d\theta}{(m^2 - 2m\cos 2\theta + 1)} = \frac{2\pi}{1-m^2},$$

$$\int \frac{\cos 2\theta \, d\theta}{(m^2 - 2m\cos 2\theta + 1)} = \frac{2\pi m}{1-m^2},$$

$$\int \frac{\cos^2 2\theta \, d\theta}{(m^2 - 2m\cos 2\theta + 1)} = \frac{\pi(1+m^2)}{1-m^2},$$

$$\int \frac{\cos^3 2\theta \, d\theta}{(m^2 - 2m\cos 2\theta + 1)} = \frac{\pi m(2+m^2)}{(1-m^2)},$$

$$\int \frac{d\theta}{(m^2 - 2m\cos 2\theta + 1)^2} = \frac{2\pi(1+m^2)}{(1-m^2)^3},$$

$$\int \frac{\cos 2\theta \, d\theta}{(m^2 - 2m\cos 2\theta + 1)^2} = \frac{4\pi m}{(1-m^2)^3},$$

$$\int \frac{\cos^2 2\theta \, d\theta}{(m^2 - 2m\cos 2\theta + 1)^2} = \frac{\pi(1+4m^2-m^4)}{(1-m^2)^3},$$

$$\int \frac{\cos^3 2\theta \, d\theta}{(m^2 - 2m\cos 2\theta + 1)^2} = \frac{\pi m(3+2m^2-m^4)}{(1-m^2)^3},$$

$$\int \frac{d\theta}{\{(a^2+h_1^2 b^2) - (a^2-h_1^2 b^2)\cos 2\theta\}} = \frac{\pi}{h_1 a b},$$

$$\int \frac{\cos 2\theta \, d\theta}{\{(a^2+h_1^2 b^2) - (a^2-h_1^2 b^2)\cos 2\theta\}} = \frac{\pi(a-h_1 b)}{h_1 a b(a+h_1 b)},$$

$$\int \frac{\cos^2 2\theta \, d\theta}{\{(a^2+h_1^2 b^2) - (a^2-h_1^2 b^2)\cos 2\theta\}} = \frac{\pi(a^2+h_1^2 b^2)}{h_1 a b(a+h_1 b)^2}.$$

APPENDIX IV

Particular Cases Referred to Chapters XI, XII

The results obtained in chapters XI and XII may be used to deduce some interesting results. We first consider the spherical case.

It is seen from equation (152), that the value of ϵ depends upon the ratio $b|a$ also. Setting this to n, we get

$$\epsilon = \frac{E_1\{n^3(1+\nu)+2(1-2\nu)\}\delta - 3\,P_0\, n^3(1-\nu)(1-2\nu_1)}{E_1\{n^3(1+\nu)+2(1-2\nu)\} + 2E\,(n^3-1)(1-2\nu_1)}.$$

If n tends to infinity,

$$\epsilon = \frac{E_1(1+\nu)\delta - 3\,P_0(1-\nu)(1-2\nu_1)}{E_1(1+\nu) + 2E\,(1-2\nu_1)}.$$

Putting $P_0 = 0$, we recover the result

$$\epsilon = \frac{E_1(1+\nu)\delta}{E_1(1+\nu)+2E\,(1-2\nu_1)},$$

given in appendix I.

If the matrix is thin, we set $n = 1+\alpha$. Neglecting the second and higher order quantities, we have

$$\epsilon = \frac{E_1(1-\nu)\{E_1\delta + P_0(1-2\nu_1)\} + 2\alpha\, P_0\{E_1(1-2\nu)-E\,(1-2\nu_1)\}(1-2\nu)}{E_1^2(1-\nu)^2}.$$

The hoop stress in the matrix at equilibrium boundary is also of interest as it indicates whether the material outside can withstand the spontaneous change of the inclusion. In this case the yielding criterion of Von Mises or of Tresca, is also governed at the boundary by the value $p_{\theta\theta} - p_{rr}$. In the case of

thin spherical shell, this value determines the ultimate strength of the shell. For the spherical thick shell, this is given by

$$p_{\theta\theta} = p_{\phi\phi} = \frac{(b^3+2a^3)EE_1\delta + 3P_0 b^3\{E_1\nu + E(1-2\nu_1)\}}{E_1\{b^3(1+\nu)+2a^3(1-2\nu)\} + 2E(b^3-a^3)(1-2\nu_1)} .$$

If the shell is a thin one,

$$p_{\theta\theta} = p_{\phi\phi} = \frac{EE_1^2\delta(1-2\nu) + P_0\{E_1\nu + E(1-2\nu_1)\}\left[E_1(1-\nu) + 2\alpha\{E_1(1-2\nu) - E(1-2\nu_1)\}\right]}{E_1^2(1-\nu)^2} .$$

As regards the cylindrical inclusion, when $b/a = n$

$$\epsilon = \frac{(\lambda_1+\mu_1)(n^2+1-2\nu)\delta + P_0 n^2(1-\nu)}{(\lambda_1+\mu_1)(n^2+1-2\nu) + \mu(n^2-1)} ,$$

$$(p_{\theta\theta})_{r=a} = \frac{\{P_0 n^2\nu + \mu(n^2+1)\delta\}(\lambda_1+\mu_1) + \mu P_0 n^2}{(\lambda_1+\mu_1)(n^2+1-2\nu) + \mu(n^2-1)} .$$

If the cylindrical shell is a thin one, such that $n = 1+\alpha$ (where powers of α higher than one are neglected), then

$$\epsilon = \frac{2(\lambda_1+\mu_1)\delta(1-\nu) + P_0\left[(\lambda_1+\mu_1)(1-\nu) + \alpha\{(\lambda_1+\mu_1)(1-2\nu)+\mu\}\right]}{2(\lambda_1+\mu_1)^2(1-\nu)^2} ,$$

$$(p_{\theta\theta})_{r=a} = \frac{P_0\{(\lambda_1+\mu_1)\nu+\mu\}\left[(\lambda_1+\mu_1)(1-\nu) + \alpha\{(\lambda_1+\mu_1)(1-2\nu)-\mu\}\right] + 2\mu(\lambda_1+\mu_1)^2(1-\nu)\delta}{2(\lambda_1+\mu_1)^2(1-\nu)^2} .$$

A simple use of these results is in the case of a solid cylinder (for example, a wheel) reinforced with a thin rim.

If the matrix is infinite, we have for the equilibrium boundary

$$\epsilon = \frac{(\lambda_1 + \mu_1)\delta + P_0(1-\nu)}{(\lambda_1 + \mu_1 + \mu)},$$

and the hoop stress at the interface is

$$(P_{\theta\theta})_{r=a} = \frac{(P_0\nu + \mu\delta)(\lambda_1 + \mu_1)}{(\lambda_1 + \mu_1 + \mu)}.$$

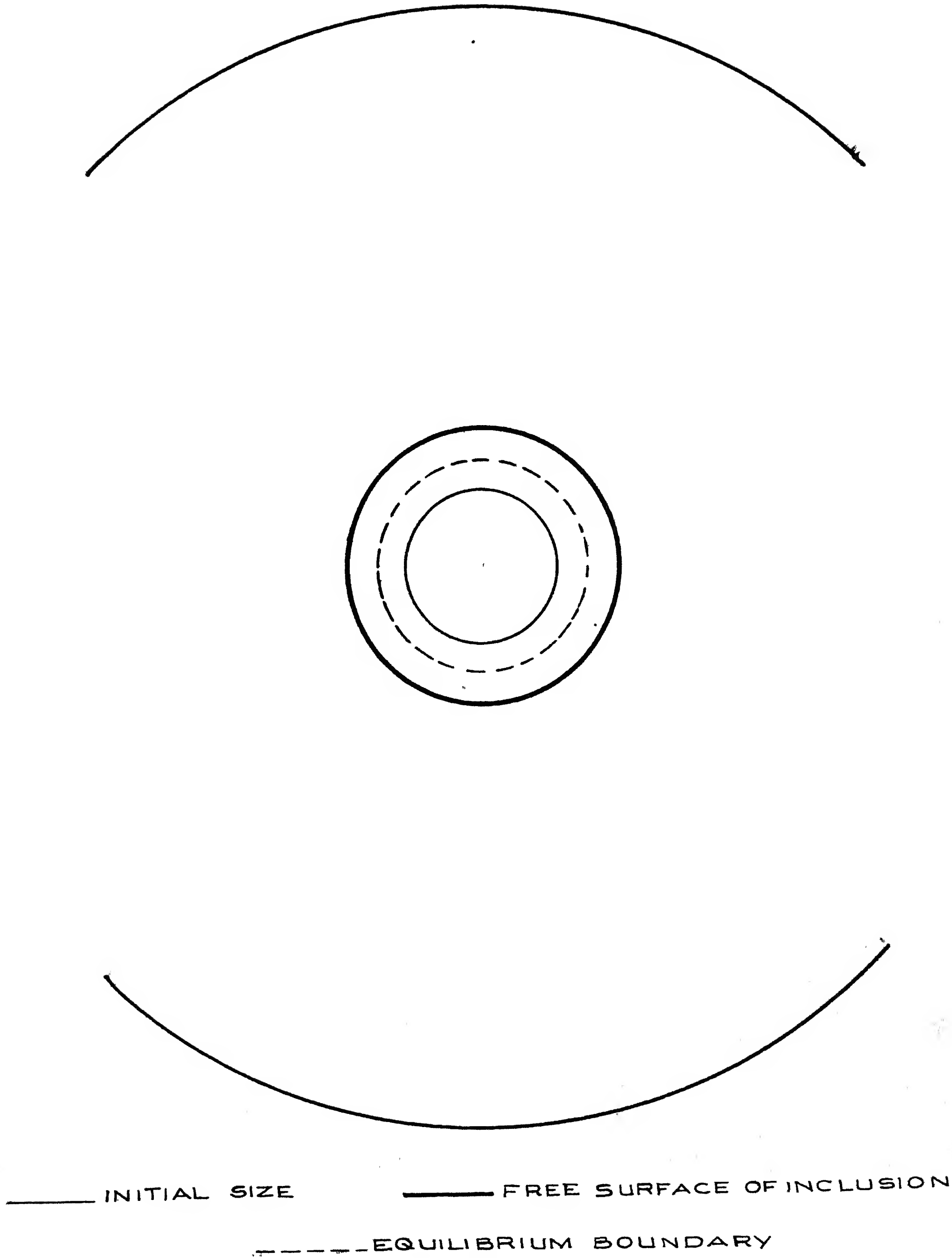

INITIAL SIZE
FREE SURFACE OF INCLUSION
EQUILIBRIUM BOUNDARY

Fig. 1.

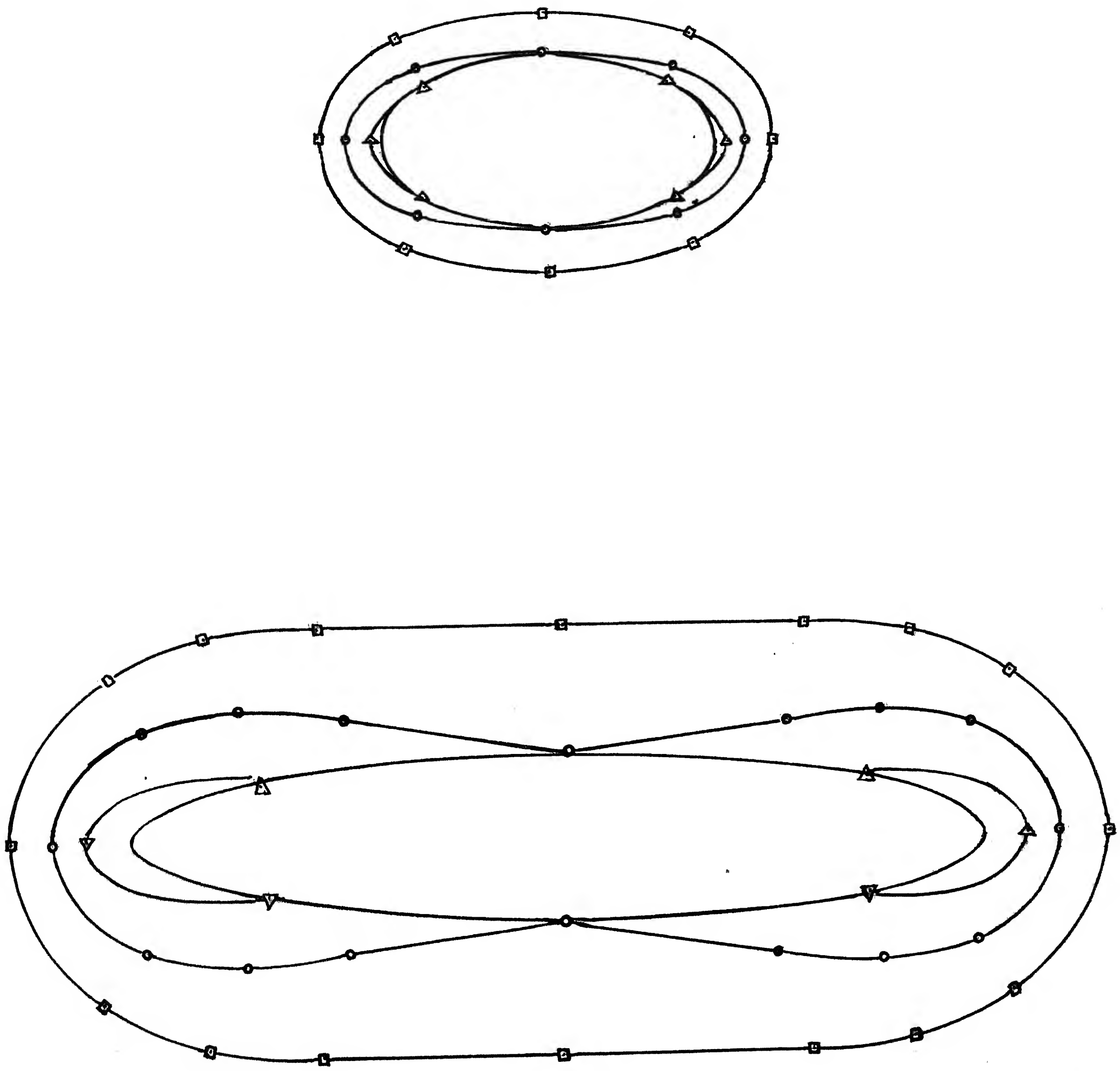
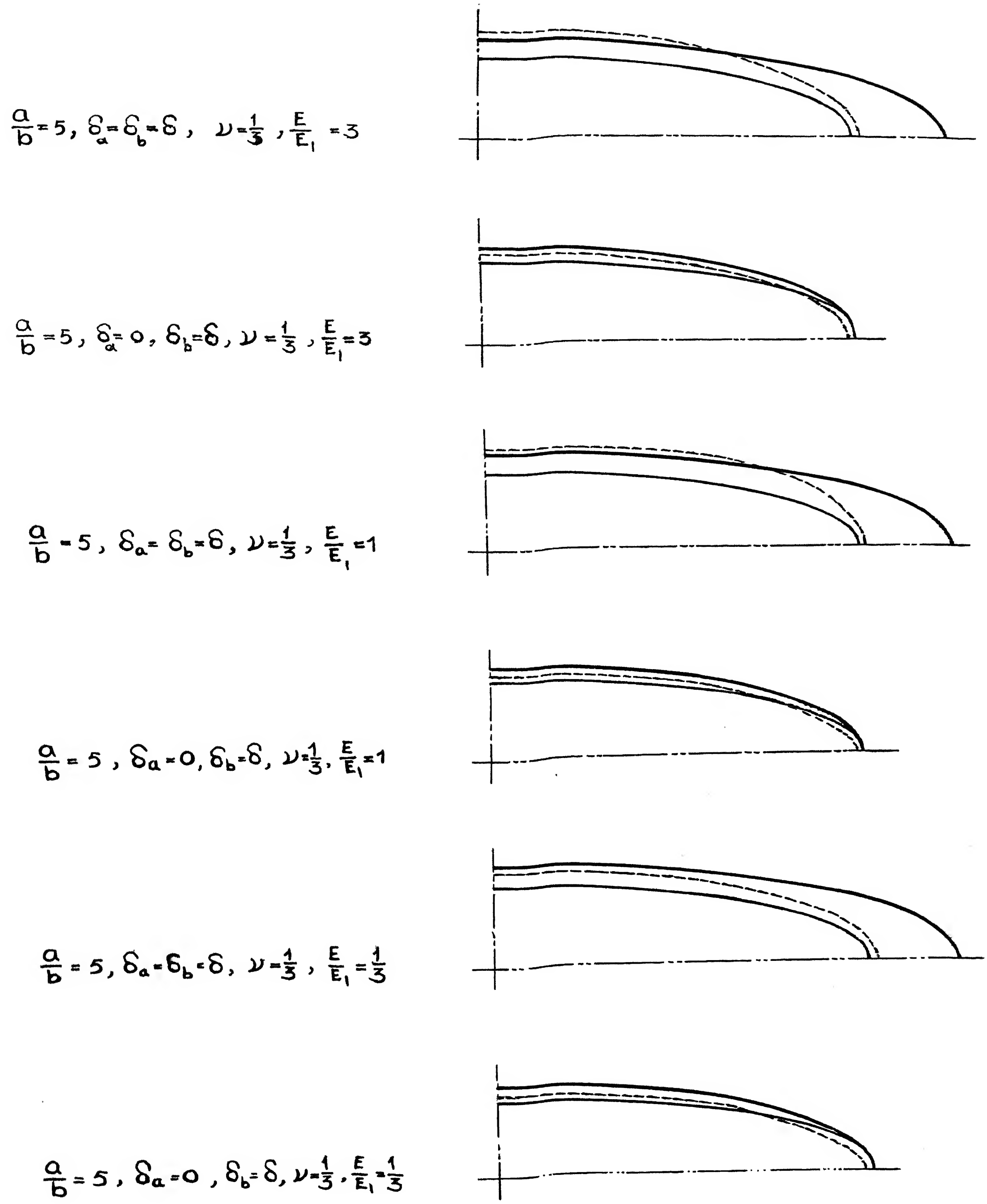
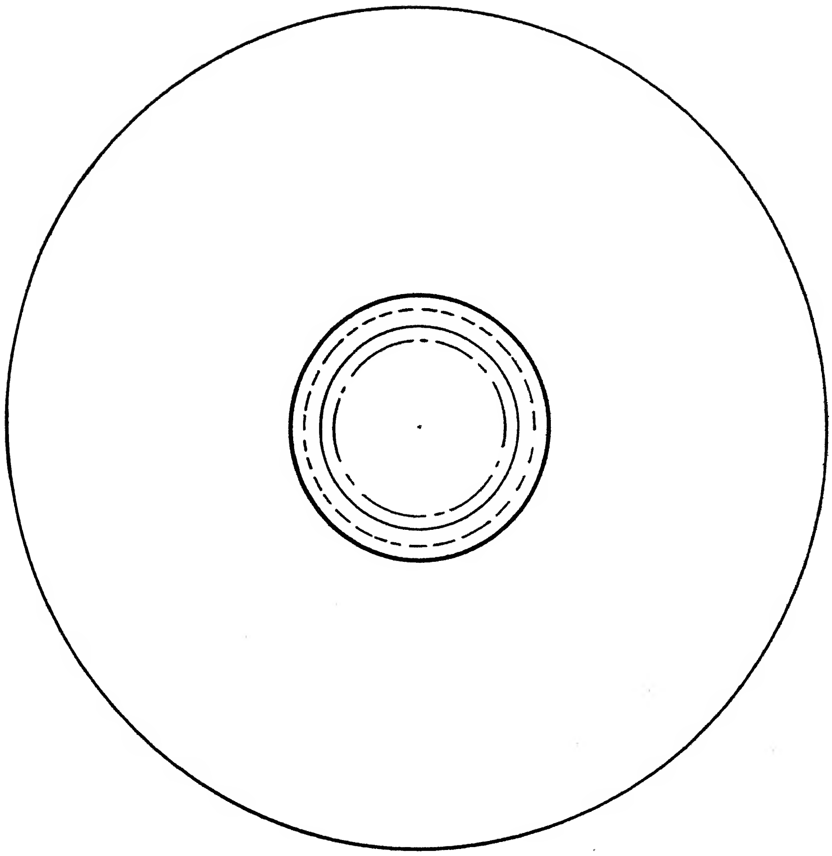


Fig. 2.



$\frac{a}{b}=5,\ \delta_a=\delta_b=\delta,\ \nu=\frac{1}{3},\ \frac{E}{E_1}=3$
$\frac{a}{b}=5,\ \delta_a=0,\ \delta_b=\delta,\ \nu=\frac{1}{3},\ \frac{E}{E_1}=3$
$\frac{a}{b}=5,\ \delta_a=\delta_b=\delta,\ \nu=\frac{1}{3},\ \frac{E}{E_1}=1$
$\frac{a}{b}=5,\ \delta_a=0,\ \delta_b=\delta,\ \nu=\frac{1}{3},\ \frac{E}{E_1}=1$
$\frac{a}{b}=5,\ \delta_a=\delta_b=\delta,\ \nu=\frac{1}{3},\ \frac{E}{E_1}=\frac{1}{3}$
$\frac{a}{b}=5,\ \delta_a=0,\ \delta_b=\delta,\ \nu=\frac{1}{3},\ \frac{E}{E_1}=\frac{1}{3}$

Fig. 3.



——— INITIAL SIZE

——— FREE SURFACE OF INCLUSION

—·—·— FREE STATE OF MATRIX

------ EQUILIBRIUM BOUNDARY

Fig. 4.

$\frac{b}{a}$	$\frac{E}{E_1}$	$\nu = \frac{1}{4}$		$\nu = \frac{1}{3}$		$\nu = \frac{1}{2}$	
		ϵ_1	ϵ_2	ϵ_1	ϵ_2	ϵ_1	ϵ_2
1	3	$\frac{11\delta_a}{50}$	$\frac{11\delta_b}{50}$	$\frac{6\delta_a+\delta_b}{14}$	$\frac{6\delta_a+\delta_b}{14}$	$\frac{5\delta_a+3\delta_b}{8}$	$\frac{5\delta_b+3\delta_a}{8}$
1	1	$\frac{\delta_a}{2}$	$\frac{\delta_b}{2}$	$\frac{11\delta_a+\delta_b}{16}$	$\frac{11\delta_b+\delta_a}{16}$	$\frac{3\delta_a+\delta_b}{4}$	$\frac{3\delta_b+\delta_a}{4}$
1	$\frac{1}{3}$	$\frac{75\delta_a}{98}$	$\frac{2(31\delta_b+3\delta_a)}{98}$	$\frac{26\delta_a+\delta_b}{30}$	$\frac{26\delta_b+\delta_a}{30}$	$\frac{7\delta_a+\delta_b}{8}$	$\frac{7\delta_b+\delta_a}{8}$
$\frac{1}{2}$	3	$\frac{35\delta_a-3\delta_b}{227}$	$\frac{2(21\delta_b+\delta_a)}{227}$	$\frac{\delta\,1}{4}$	$\frac{10\delta_b+3\delta_a}{16}$	$\frac{7\delta_a+3\delta_b}{19}$	$\frac{4(4\delta_b+3\delta_a)}{19}$
$\frac{1}{2}$	1	$\frac{33\delta_a-\delta_b}{81}$	$\frac{2(21\delta_b+\delta_a)}{81}$	$\frac{\delta_a}{2}$	$\frac{5\delta_b+\delta_a}{6}$	$\frac{5\delta_a+\delta_b}{9}$	$\frac{4(2\delta_b+\delta_a)}{9}$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3(97\delta_a-\delta_b)}{419}$	$\frac{6(21\delta_b+\delta_a)}{419}$	$\frac{3\delta_a}{4}$	$\frac{3(10\delta_b+\delta_a)}{32}$	$\frac{13\delta_a+\delta_b}{17}$	$\frac{4(4\delta_b+\delta_a)}{17}$
$\frac{1}{5}$	3	$\frac{37\delta_a-18\delta_b}{466}$	$\frac{5(29\delta_b+18\delta_a)}{446}$	$\frac{14\delta_a-3\delta_b}{134}$	$\frac{5(22\delta_b+9\delta_a)}{134}$	$\frac{13\delta_a+\delta_b}{88}$	$\frac{5(17\delta_b+15\delta_a)}{88}$
$\frac{1}{5}$	1	$\frac{13\delta_a-2\delta_b}{54}$	$\frac{5(5\delta_b+2\delta_a)}{54}$	$\frac{13\delta_a-\delta_b}{48}$	$\frac{5(3\delta_b+\delta_a)}{16}$	$\frac{11\delta_a+\delta_b}{36}$	$\frac{5(7\delta_b+5\delta_a)}{36}$
$\frac{1}{5}$	$\frac{1}{3}$	$\frac{3(119\delta_a-6\delta_b)}{706}$	$\frac{15(31\delta_b+6\delta_a)}{706}$	$\frac{3(38\delta_a-\delta_b)}{214}$	$\frac{5(14\delta_b+\delta_a)}{214}$	$\frac{31\delta_a+\delta_b}{56}$	$\frac{5(11\delta_b+5\delta_a)}{56}$
$\frac{1}{10}$	3	$\frac{139\delta_a-81\delta_b}{3187}$	$\frac{10(103\delta_b+81\delta_a)}{3187}$	$\frac{3\delta_a-\delta_b}{58}$	$\frac{5(42\delta_b+19\delta_a)}{232}$	$\frac{23\delta_a+3\delta_b}{353}$	$\frac{20(16\delta_b+15\delta_a)}{121}$
$\frac{1}{10}$	1	$\frac{153\delta_a-27\delta_b}{1089}$	$\frac{10(45\delta_b+27\delta_a)}{1087}$	$\frac{537\delta_a-4\delta_b}{242}$	$\frac{5(47\delta_b+19\delta_a)}{242}$	$\frac{21\delta_a+\delta_b}{121}$	$\frac{20(8\delta_b+5\delta_a)}{161}$
$\frac{1}{10}$	$\frac{1}{3}$	$\frac{3(473\delta_a-27\delta_b)}{4147}$	$\frac{30(77\delta_b+27\delta_a)}{4147}$	$\frac{28\delta_a-\delta_b}{78}$	$\frac{5(62\delta_b+19\delta_a)}{312}$	$\frac{61\delta_a+\delta_b}{161}$	$\frac{20(8\delta_b+5\delta_a)}{161}$
$\frac{1}{\infty}$	$\frac{E}{E_1}$	0	$\frac{\delta_a+\delta_b}{3}$	0	$\frac{2\delta_b+\delta_a}{3}$	0	$(\delta_a+\delta_b)$
$\frac{b}{a}$	0	δ_a	δ_b	δ_a	δ_b	δ_a	δ_b
$\frac{b}{a}$	∞	0	0	0	0	0	0

Table.1.

$\frac{b}{a}$	$\frac{E}{E_1}$	$\nu = \frac{1}{4}$		$\nu = \frac{1}{3}$		$\nu = \frac{1}{2}$	
		$\frac{r_1}{r_a + r_b}$	$\frac{r_2}{r_a + r_b}$	$\frac{r_1}{r_a + r_b}$	$\frac{r_2}{r_a + r_b}$	$\frac{r_1}{r_a + r_b}$	$\frac{r_2}{r_a + r_b}$
1	3	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{5}{28}$	$\frac{5}{28}$	$\frac{1}{8}$	$\frac{1}{8}$
1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{3}$	$\frac{9}{7}$	$\frac{9}{7}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{3}{8}$
$\frac{1}{2}$	3	$\frac{14}{43}$	$\frac{5}{43}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{4}{17}$	$\frac{1}{17}$
$\frac{1}{2}$	1	$\frac{14}{27}$	$\frac{5}{27}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{4}{9}$	$\frac{1}{9}$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{42}{65}$	$\frac{3}{13}$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{12}{19}$	$\frac{3}{19}$
$\frac{1}{5}$	3	$\frac{20}{43}$	$\frac{2}{43}$	$\frac{35}{68}$	$\frac{3}{68}$	$\frac{25}{56}$	$\frac{1}{56}$
$\frac{1}{5}$	1	$\frac{20}{27}$	$\frac{2}{27}$	$\frac{35}{48}$	$\frac{1}{16}$	$\frac{25}{36}$	$\frac{1}{36}$
$\frac{1}{5}$	$\frac{1}{3}$	$\frac{12}{13}$	$\frac{6}{65}$	$\frac{105}{124}$	$\frac{9}{124}$	$\frac{75}{88}$	$\frac{3}{88}$
$\frac{1}{10}$	3	$\frac{310}{443}$	$\frac{11}{443}$	$\frac{205}{302}$	$\frac{7}{302}$	$\frac{100}{161}$	$\frac{1}{161}$
$\frac{1}{10}$	1	$\frac{310}{363}$	$\frac{1}{33}$	$\frac{205}{242}$	$\frac{7}{242}$	$\frac{100}{121}$	$\frac{1}{121}$
$\frac{1}{10}$	$\frac{1}{3}$	$\frac{630}{1009}$	$\frac{33}{1009}$	$\frac{205}{222}$	$\frac{7}{222}$	$\frac{300}{323}$	$\frac{3}{323}$
$\frac{1}{\infty}$	$\frac{E}{E_1}$	$r_a + r_b$	0	$r_a + r_b$	0	$r_a + r_b$	0
$\frac{b}{a}$	∞	0	0	0	0	0	0

Table 2.

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